Chapter 18

Measurements II

18.1 Beam Splitter and Successive Measurements

Sometimes a quantum system, hereafter referred to as a "particle", is destroyed during a measurement process, but in other cases it continues to exist in an identifiable form after interacting with the measuring apparatus, with some of its properties unchanged or related in a non-trivial way to properties which it possessed before this interaction. In such a case it is interesting to ask what will happen if a second measurement is carried out on the particle: how will the outcome of the second measurement be related to the outcome of the first measurement, and to properties of the particle between the two measurements?

Let us consider a specific example in which a particle (photon or neutron) passes through a beam splitter B and is then subjected to a measurement by nondestructive detectors located in the c and d output channels as shown in Fig. 18.1. Assume that the unitary time development of the particle in the absence of any measuring devices is given by

$$|0a\rangle \mapsto \big(|1c\rangle + |1d\rangle\big)/\sqrt{2} \mapsto \big(|2c\rangle + |2d\rangle\big)/\sqrt{2} \mapsto \cdots \tag{18.1}$$

as time progresses from t_0 to t_1 to $t_2 \ldots$. Here the kets denote wave packets whose approximate locations are shown by the circles in Fig. 18.1, and the labels are similar to those used for the toy model in Ch. 12.

The detectors are assumed to register the passage of the particle while having a negligible influence on the time development of its wave packet. Toy detectors with this property were introduced earlier, in Secs. 7.4 and 12.3. To actually construct such a device in the laboratory is much more difficult, but, at least for some types of particle, not out of the question. We assume that the interaction of the particle with the detector C in Fig. 18.1 leads to a unitary time development during the interval from t_1 to t_2 of the form

$$\begin{aligned} |1c\rangle|C^\circ\rangle &\mapsto |2c\rangle|C^*\rangle, \\ |1d\rangle|C^\circ\rangle &\mapsto |2d\rangle|C^\circ\rangle, \end{aligned} \tag{18.2}$$

where $|C^\circ\rangle$ denotes the "ready" or "untriggered" state of the detector, and $|C^*\rangle$ the "triggered" state orthogonal to $|C^\circ\rangle$. (The tensor product symbol, as in $|1c\rangle \otimes |C^\circ\rangle$, has been omitted.) The

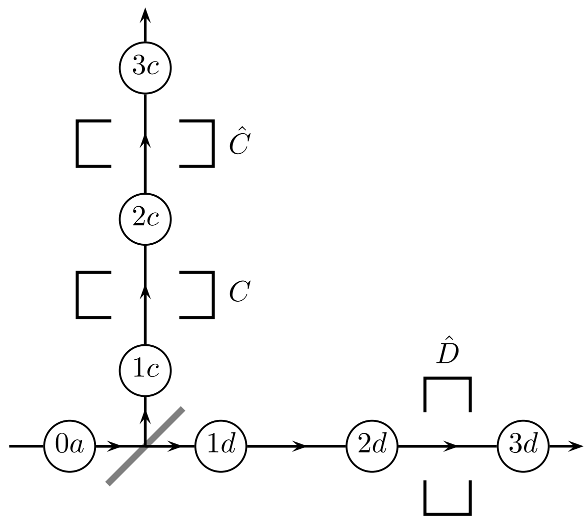

Figure 18.1: Beam splitter followed by nondestructive measuring devices. The circles indicate the locations of wave packets corresponding to different kets.

behavior of the other detectors $\hat{C}$ and $\hat{D}$ is similar, and thus an initial state

$$|\Psi_0\rangle = |0a\rangle|C^\circ\rangle|\hat{C}^\circ\rangle|\hat{D}^\circ\rangle \tag{18.3}$$

develops unitarily to

$$|\Psi_1\rangle = \big(|1c\rangle + |1d\rangle\big)|C^\circ\rangle|\hat{C}^\circ\rangle|\hat{D}^\circ\rangle/\sqrt{2}, \tag{18.4}$$

$$|\Psi_2\rangle = \big(|2c\rangle|C^*\rangle|\hat{C}^\circ\rangle|\hat{D}^\circ\rangle + |2d\rangle|C^\circ\rangle|\hat{C}^\circ\rangle|\hat{D}^\circ\rangle\big)/\sqrt{2}, \tag{18.5}$$

$$|\Psi_3\rangle = \big(|3c\rangle|C^*\rangle|\hat{C}^*\rangle|\hat{D}^\circ\rangle + |3d\rangle|C^\circ\rangle|\hat{C}^\circ\rangle|\hat{D}^*\rangle\big)/\sqrt{2} \tag{18.6}$$

at the times t_1, t_2, t_3.

We shall be interested in families of histories based on the initial state $|\Psi_0\rangle$. The simplest one to understand in physical terms is a family $\mathcal{F}$ in which at the times t_1, t_2, and t_3 every detector is either ready or triggered, and the particle is represented by a wave packet in one of the two output channels. The support of $\mathcal{F}$ consists of the two histories

$$\begin{aligned} Y^c &= \Psi_0 \odot [1c]C^\circ\hat{C}^\circ\hat{D}^\circ \odot [2c]C^*\hat{C}^\circ\hat{D}^\circ \odot [3c]C^*\hat{C}^*\hat{D}^\circ, \\ Y^d &= \Psi_0 \odot [1d]C^\circ\hat{C}^\circ\hat{D}^\circ \odot [2d]C^\circ\hat{C}^\circ\hat{D}^\circ \odot [3d]C^\circ\hat{C}^\circ\hat{D}^*, \end{aligned} \tag{18.7}$$

where square brackets have been omitted from $[\Psi_0]$, $[C^\circ]$, etc., so that the formula remains valid if one employs macro projectors, as in Sec. 17.4. In Y^c the particle moves out along channel c and triggers the detectors C and $\hat{C}$ as it passes through them, while in Y^d the particle moves along channel d and triggers $\hat{D}$.

The situation described by these histories is essentially the same as it would be if a *classical* particle were scattered at random by the beam splitter into either the c or the d channel, and then traveled out along the channel triggering the corresponding detector(s). Thus if C is triggered at time t_2 the particle is surely in the c channel, and will later trigger $\hat{C}$, whereas if C is still in its ready state at t_2, this means the particle is in the d channel, and will later trigger $\hat{D}$. That these assertions are indeed correct for a quantum particle can be seen by working out various conditional probabilities, e.g.

$$\Pr([1c]_1 \,|\, C_2^*) = 1 = \Pr([1d]_1 \,|\, C_2^\circ), \tag{18.8}$$

$$\Pr([2c]_2 \,|\, C_2^*) = 1 = \Pr([2d]_2 \,|\, C_2^\circ), \tag{18.9}$$

$$\Pr([3c]_3 \,|\, C_2^*) = 1 = \Pr([3d]_3 \,|\, C_2^\circ), \tag{18.10}$$

where the subscripts indicate the times, t_1 or t_2 or t_3, at which the events occur. Thus the location of the particle either before or after t_2 can be inferred from whether it has or has not been detected by C at t_2. There are, in addition, correlations between the outcomes of the different measurements:

$$\Pr(\hat{C}_3^* \,|\, C_2^*) = 1, \quad \Pr(\hat{D}_3^* \,|\, C_2^*) = 0, \tag{18.11}$$

$$\Pr(\hat{C}_3^\circ \,|\, C_2^\circ) = 1, \quad \Pr(\hat{D}_3^* \,|\, C_2^\circ) = 1. \tag{18.12}$$

Thus whether $\hat{C}$ or $\hat{D}$ will later detect the particle is determined by whether it was or was not detected earlier by C.

The conditional probabilities in (18.8) to (18.12) are straightforward consequences of the fact that all histories in $\mathcal{F}$ except for the two in (18.7) have zero probability. Since these conditional probabilities, with the exception of (18.9), involve more than two times—note that the initial Ψ_0 is implicit in the condition—they cannot be obtained by using the Born rule, and are therefore inaccessible to older approaches to quantum theory which lack the formalism of Ch. 10. These older approaches employ a notion of "wave function collapse" in order to get results comparable to (18.9) through (18.12), and this is the subject of the next section.

18.2 Wave Function Collapse

Quantum measurements have often been analyzed using the following idea, which goes back to von Neumann. Consider an isolated system $\mathcal{S}$, and suppose that its wave function evolves unitarily, so that it is $|s_1\rangle$ at a time t_1. At this time, or very shortly thereafter, $\mathcal{S}$ interacts with a measuring apparatus $\mathcal{M}$ designed to determine whether $\mathcal{S}$ is in one of the states of a collection $\{|s^k\rangle\}$ forming an orthonormal basis of the Hilbert space of $\mathcal{S}$. The measurement will have an outcome k with probability $|\langle s_1|s^k\rangle|^2$, and if the outcome is k the effect of the measurement will be to "collapse" or "reduce" $|s_1\rangle$ to $|s^k\rangle$.

This collapse picture of a measurement proceeds in the following way when $\mathcal{S}$ is the particle and $\mathcal{M}$ the detector C in Fig. 18.1. The particle undergoes unitary time evolution until it encounters the measuring apparatus, and thus at t_1 it is in a state

$$|1a\rangle = (|1c\rangle + |1d\rangle)/\sqrt{2}. \tag{18.13}$$

The detector at time t_2 is either still in its ready state $|C^\circ\rangle$, or else in its triggered state $|C^*\rangle$ indicating that it has detected the particle. If the particle has been detected, its wave function will have collapsed from its earlier delocalized state $|1a\rangle$ into the $|2c\rangle$ wave packet localized in the c channel and moving towards detector $\hat{C}$, which will later detect the particle. If, on the other hand, the particle has *not* been detected by C, its wave function will have collapsed into the packet $|2d\rangle$ localized in the d channel and moving towards the $\hat{D}$ detector, which will later register the passage of the particle. Consequently C^* at t_2 results in $\hat{C}^*$ at t_3, whereas C° at t_2 implies $\hat{D}^*$ at t_3, in agreement with the conditional probabilities in (18.11) and (18.12).

This "collapse" picture has long been regarded by many quantum physicists as rather unsatisfactory for a variety of reasons, among them the following. First, it seems somewhat arbitrary to abandon the state $|\Psi_2\rangle$ obtained by unitary time evolution, (18.5), without providing some better reason than the fact that a measurement occurs; after all, what is special about a quantum measurement? All real measurement apparatus is constructed out of aggregates of particles to which the laws of quantum mechanics apply, so the apparatus ought to be described by those laws, and not used to provide an excuse for their breakdown. Second, while it might seem plausible that an interaction sufficient to trigger a measuring apparatus could somehow localize a particle wave packet somewhere in the vicinity of the apparatus, it is much harder to understand how the same apparatus by *not* detecting the particle manages to localize it in some region which is very far away.

This second, nonlocal aspect of the collapse picture is particularly troublesome, and has given rise to an extensive discussion of "interaction-free measurements" in which some property of a quantum system can be inferred from the fact that it did *not* interact with a measuring device. Since one can imagine the gedanken experiment in Fig. 18.1 set up in outer space with the wave packets $|2c\rangle$ and $|2d\rangle$ an enormous distance apart, there is also the problem that if wave function collapse takes place instantaneously it will conflict with the principle of special relativity according to which no influence can travel faster than the speed of light.

By contrast, the analysis given above in Sec. 18.1 based upon the family $\mathcal{F}$, (18.7), shows no signs of any nonlocal effects. If C has not detected the particle at time t_2, this is because the particle is moving out the d channel, not the c channel. In the case of a classical particle such an "interaction free measurement" of the channel in which it is moving gives rise to no conceptual difficulties or conflicts with relativity theory. As pointed out in Sec. 18.1, the family $\mathcal{F}$ provides a quantum description which resembles that of a classical particle, and thus by using this family one avoids the nonlocality difficulties of wave function collapse.

Another way to avoid these difficulties is to think of wave function collapse not as a

physical effect produced by the measuring apparatus, but as a *mathematical procedure* for calculating statistical correlations of the type shown in (18.9) to (18.12). That is, "collapse" is something which takes place in the theorist's notebook, rather than the experimentalist's laboratory. In particular, if the wave function is thought of as a pre-probability (Sec. 9.4), then it is perfectly reasonable to collapse it to a different pre-probability in the middle of a calculation.

With reference to the arrangement in Fig. 18.1, the idea of wave function collapse corresponds fairly closely to a consistent family $\mathcal{V}$ with support

$$\Psi_0 \odot \Psi_1 \odot \begin{cases} [2c]C^*\hat{C}^\circ\hat{D}^\circ \odot [3c]C^*\hat{C}^*\hat{D}^\circ, \\ [2d]C^\circ\hat{C}^\circ\hat{D}^\circ \odot [3d]C^\circ\hat{C}^\circ\hat{D}^*. \end{cases} \tag{18.14}$$

These two histories represent unitary time evolution of the initial state, so they are identical up to the time t_1, before the particle interacts (or fails to interact) with C, but are thereafter distinct.

As a consequence of the internal consistency of quantum reasoning, Sec. 16.3, this family gives the same results for the conditional probabilities in (18.9) to (18.12) as does $\mathcal{F}$. (Those in (18.8) are not defined in $\mathcal{V}$.) In particular, either family can be used to predict the outcomes of later $\hat{C}$ and $\hat{D}$ measurements based upon the outcome of the earlier C measurement.

One can imagine constructing the framework $\mathcal{V}$ in successive steps as follows. Use unitary time development up to t_2, but think of $|\Psi_2\rangle$ in (18.5) as a pre-probability (rather than as representing an MQS property) useful for assigning probabilities to the two histories

$$\Psi_0 \odot \Psi_1 \odot \{C^*, C^\circ\}, \tag{18.15}$$

which form the support of a consistent family whose projectors at t_2 represent the two possible measurement outcomes. This is the minimum modification of a unitary family which can exhibit these outcomes. Next refine this family by including the corresponding particle properties at t_2 along with the ready states of the other detectors:

$$\Psi_0 \odot \Psi_1 \odot \begin{cases} [2c]C^*\hat{C}^\circ\hat{D}^\circ, \\ [2d]C^\circ\hat{C}^\circ\hat{D}^\circ. \end{cases} \tag{18.16}$$

Finally, use unitary extensions of these histories, Sec. 11.7, to obtain the family $\mathcal{V}$ of (18.14). In a more general situation the step from (18.15) to (18.16) can be more complicated: one may need to use conditional density matrices rather than projectors onto particle properties, as discussed in Sec. 15.7. But the general idea is still the same: information from the outcome of a measurement is used to construct a new initial state of the particle, which is then employed for calculating results at still later times. Wave function collapse is, in essence, an algorithm for constructing this new initial state given the outcome of the measurement.

Wave function collapse is in certain respects analogous to the "collapse" of a classical probability distribution when it is conditioned on the basis of new information. Once again think of a classical particle randomly scattered by the beam splitter into the c or d channel. Before the particle (possibly) passes through C, it is delocalized in the sense that the probability is $1/2$ for it to be in either the c or the d channel. But when the probability for the location of the particle is conditioned on the measurement outcome it collapses in the sense that the particle is either in the c channel, given C^*, or the d channel, given C°. This collapse of the classical probability distribution is obviously not a physical effect, and only in some metaphorical sense can it be said to be "caused" by the measurement. This becomes particularly clear when one notes that conditioning on the measurement outcome collapses the probability distribution at a time t_1 *before* the measurement occurs in the same way that it collapses it at t_2 or t_3 *after* the measurement occurs. Thinking of the collapse as being caused by the measurement would lead to an odd situation in which an effect precedes its cause.

Precisely the same comment applies to the collapse of a quantum wave function. A quantum description conditioned on a particular outcome of a measurement will generally provide more detail, and thus appear to be "collapsed", in comparison with one constructed without this information. But since the outcome of a quantum measurement can also tell one something about the properties of the measured particle prior to the measurement process (assuming a framework in which these properties can be discussed) one should not think of the collapse as some sort of physical effect with a physical cause. To be sure, in the family (18.14) it is not possible to discuss the location (c or

d) of the particle before the measurement, because in this particular framework the location does not make sense. The implicit use of this type of family for discussions of quantum measurements is probably one reason why wave function collapse has often been confused with a physical effect. The availability of other families, such as $\mathcal{F}$ in (18.7), helps one avoid this mistake.

In summary, when quantum mechanics is formulated in a consistent way, wave function collapse is not needed in order to describe the interaction between a particle (or some other quantum system) and a measuring device. One can use a notion of collapse as a method of constructing a particular type of consistent family, as indicated in the steps leading from (18.15) to (18.16) to (18.14), or else as a picturesque way of thinking about correlations that in the more sober language of ordinary probability theory are written as conditional probabilities, as in (18.9) to (18.12). However, for neither of these purposes is it actually essential; any result that can be obtained by collapsing a wave function can also be obtained in a straightforward way by adopting an appropriate family of histories. The approach using histories is more flexible, and allows one to describe the measurement process in a natural way as one in which the properties of the particle before as well as after the measurement are correlated to the measurement outcomes.

While its picturesque language may have some use for pedagogical purposes or for constructing mnemonics, the concept of wave function collapse has given rise to so much confusion and misunderstanding that it would, in my opinion, be better to abandon it altogether, and instead use conditional states, such as the conditional density matrices discussed in Sec. 15.7 and in Sec. 18.5 below, and conditional probabilities. These are quite adequate for constructing quantum descriptions, and are much less confusing.

18.3 Nondestructive Stern-Gerlach Measurements

The Stern-Gerlach apparatus for measuring one component of spin angular momentum of a spin-half atom was described in Ch. 17. Here we shall consider a modified version which, although it would be extremely difficult to construct in the laboratory, does not violate any principles of quantum mechanics, and is useful for understanding why quantum measurements that are nondestructive for certain properties will be destructive for other properties. Figure 18.2 shows the modified apparatus, which consists of several parts. First, a magnet with an appropriate field gradient like the one in Fig. 17.1 separates the incoming beam into two diverging beams depending upon the value of S_z, with the $S_z = +1/2$ beam going upwards and the $S_z = -1/2$ beam going downwards. There are then two additional magnets, with field gradients in a direction opposite to the gradient in the first magnet, to bend the separated beams in such a way that they are traveling parallel to each other. These beams pass through detectors D_a and D_b of the nondestructive sort employed in Fig. 18.1. We assume not only that these detectors produce a negligible perturbation of the spatial wave packets in each beam, but also that they do not perturb the z component of spin. (A detector in one beam and not the other would actually be sufficient, but using two emphasizes the symmetry of the situation.) The detectors are followed by a series of magnets which reverse the process produced by the first set of magnets and bring the two beams back together again.

The net result is that an atom with either $S_z = +1/2$ or $S_z = -1/2$ will traverse the apparatus and emerge in the same beam at the other end. The only difference is in the detector which is triggered while the atom is inside the apparatus. The unitary time evolution corresponding to the

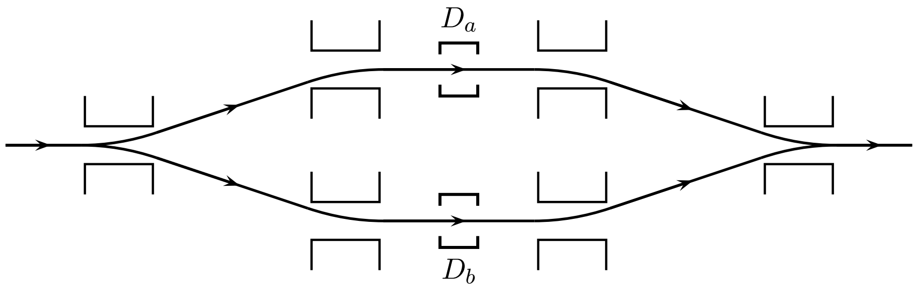

Figure 18.2: Modified Stern-Gerlach apparatus for nondestructive measurements of S_z.

measurement process is

$$|z^+\rangle|Z^\circ\rangle \mapsto |z^+\rangle|Z^+\rangle, \quad |z^-\rangle|Z^\circ\rangle \mapsto |z^-\rangle|Z^-\rangle, \tag{18.17}$$

where $|z^\pm\rangle$ are spin states corresponding to $S_z = \pm 1/2$, $|Z^\circ\rangle$ is the initial state of the apparatus, and $|Z^+\rangle$ and $|Z^-\rangle$ are mutually orthogonal apparatus states corresponding to detection by the upper or by the lower detector in Fig. 18.2. One could equally well use macro projectors for the apparatus states, as in Sec. 17.4, and for this reason we will employ Z° and $Z^\pm$ without square brackets as symbols for the corresponding projectors. In addition, the coordinate representing the center of mass of the atom is not shown in (18.17); omitting it will cause no confusion, and including it would merely clutter the notation. We shall assume that there are no magnetic fields outside the apparatus which could affect the atom's spin, and that the apparatus states $|Z^\circ\rangle$ and $|Z^\pm\rangle$ do not change with time except when interacting with the atom, (18.17). The latter assumption is convenient, but not essential.

It is obvious that the same type of apparatus can be used to measure other components of spin by using a different direction for the magnetic field gradient. For example, if the atom is thought of as moving along the y axis, then by simply rotating the apparatus about this axis it can be used to measure S_w for w any direction in the x, z plane. Alternatively, one could arrange for the atom to pass through regions of uniform magnetic field before entering and after leaving the apparatus sketched in Fig. 18.2, in order to cause a precession of an atom with $S_w = \pm 1/2$ into one with $S_z = \pm 1/2$, and then back again after the measurement is over.

We will consider various histories based upon an initial state

$$|\Psi_0\rangle = |u^+\rangle|Z^\circ\rangle, \tag{18.18}$$

at the time t_0, where the kets

$$\begin{aligned} |u^+\rangle &= +\cos(\vartheta/2)|z^+\rangle + \sin(\vartheta/2)|z^-\rangle, \\ |u^-\rangle &= -\sin(\vartheta/2)|z^+\rangle + \cos(\vartheta/2)|z^-\rangle, \end{aligned} \tag{18.19}$$

see (4.14), correspond to $S_u = +1/2$ and $-1/2$ for a direction u in the x, z plane at an angle ϑ to the $+z$ axis, so that S_u is equal to S_z when $\vartheta = 0$, and S_x when $\vartheta = \pi/2$.

Consider the consistent family with support

$$\Psi_0 \odot \begin{cases} [z^+] \odot Z^+ \odot [z^+], \\ [z^-] \odot Z^- \odot [z^-], \end{cases} \tag{18.20}$$

where the projectors refer to an initial time t_0, a time t_1 before the atom enters the apparatus, a time t_2 when it has left the apparatus, and a still later time t_3. The conditional probabilities

$$\Pr([z^+]_1 \,|\, Z_2^+) = 1 = \Pr([z^-]_1 \,|\, Z_2^-) \tag{18.21}$$

show that the properties $S_z = \pm 1/2$ before the measurement are correlated with the measurement outcomes, so that the apparatus does indeed carry out a measurement. In addition, the probabilities

$$\begin{aligned} \Pr([z^+]_3 \,|\, [z^+]_1) = 1 = \Pr([z^-]_3 \,|\, [z^-]_1), \\ \Pr([z^-]_3 \,|\, [z^+]_1) = 0 = \Pr([z^+]_3 \,|\, [z^-]_1) \end{aligned} \tag{18.22}$$

show that the measurement process carried out by this apparatus is nondestructive for the properties $[z^+]$ and $[z^-]$: they have the same values after the measurement as before.

Next consider a different family whose support consists of the four histories

$$\Psi_0 \odot [u^+] \odot \begin{cases} Z^+ \odot \{[u^+], [u^-]\}, \\ Z^- \odot \{[u^+], [u^-]\}. \end{cases} \tag{18.23}$$

Despite the fact that the four final projectors at t_3 are not all orthogonal to one another, the orthogonality of Z^+ and Z^- ensures consistency. It is straightforward to work out the weights associated with the different histories in (18.23) using the method of chain kets, Sec. 11.6. One result is

$$\begin{aligned} &\Pr([u^+]_3 \,|\, [u^+]_1) = |\langle u^+|z^+\rangle\langle z^+|u^+\rangle|^2 + |\langle u^+|z^-\rangle\langle z^-|u^+\rangle|^2 \\ &= (\cos(\vartheta/2))^4 + (\sin(\vartheta/2))^4 = (1+\cos^2\vartheta)/2. \end{aligned} \tag{18.24}$$

Except for $\vartheta = 0$ or π, the probability of $[u^+]$ at t_3 is less than one, meaning that the property $S_u = +1/2$ has a certain probability of being altered when the atom interacts with the apparatus designed to measure S_z. The disturbance is a maximum for $\vartheta = \pi/2$, which corresponds to $S_u = S_x$: indeed, the value of S_x after the atom has passed through the device is completely random, independent of its earlier value.

18.4 Measurements and Incompatible Families

As noted in Sec. 16.4, the relationship of incompatibility between quantum frameworks does not have a good classical analog, and thus it has to be understood in quantum mechanical terms and illustrated through quantum examples. Quantum measurements can provide useful examples, and in this section we consider two: one uses a beam splitter as in Sec. 18.1, the other employs nondestructive Stern-Gerlach devices of the type described in Sec. 18.3.

Think of a beam splitter, Fig. 18.3(a), similar to that in Fig. 18.1 except that there are no measuring devices in the output channels c and d. There is a consistent family whose support consists of the pair of histories

$$[0a] \odot \{[1c], [1d]\} \tag{18.25}$$

at the times t_0 and t_1, where the notation is the same as in Sec. 18.1. The unitary time development in (18.1) implies that each history has a probability of 1/2.

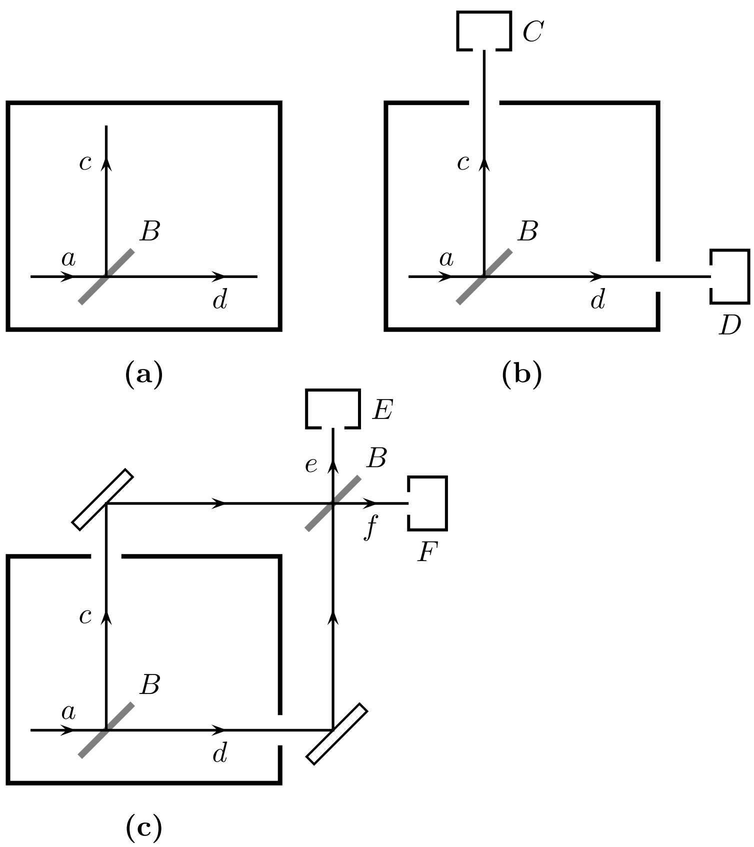

Figure 18.3: Beam splitter inside closed box (a), with two possibilities (b) and (c) for a measurement if the particle is allowed to emerge through holes in the sides of the box.

The closed box surrounding the apparatus in Fig. 18.3(a) means that we are thinking of it as an isolated quantum system. Because it is isolated, there is no direct way to check the probabilities associated with the family in (18.25). However, there is a strategy which can provide indirect evidence. Suppose that at some time later than t_1 and just before the particle would collide with one of the walls of the box, two holes are opened, as shown in Fig. 18.3(b), allowing the particle to escape and be detected by one of the two detectors C and D. If the particle is detected by C, it seems plausible that it was earlier traveling outwards through the c and not the d channel; similarly, detection by D indicates that it was earlier in the d channel. Data obtained by repeating the experiment a large number of times can be used to check that each history in (18.25) has a probability of 1/2.

Could it be that opening the box along with the subsequent measurements perturbs the particle in such a way as to invalidate the preceding analysis? This is a perfectly legitimate question, one which could also come up when one opens a "classical" box in order to determine what is going on inside it: think of a box containing unexposed photographic film, or a compressed gas. While there is no way of addressing the classical box-opening problem in a manner fully acceptable to a sceptical philosopher, a physicist will be content if he is able to achieve some reasonable understanding of what is likely to be going on during the opening process. This may require auxiliary experiments, mathematical modeling, and a certain amount of theoretical reasoning. On the basis of these a physicist might be reasonably confident when inferring something about the state of affairs inside a box before it is opened, using information from observations carried out afterwards.

Given the internal consistency of quantum reasoning, and the fact that quantum principles have been verified time and time again in innumerable experiments, it is not unreasonable to use quantum theory itself in order to examine what will happen if holes are opened in the box in Fig. 18.3, and whether the detection of the particle by C is a good reason to suppose that it was in the c channel at t_1. Carrying out such an analysis is not difficult if one assumes, as is plausible, that a timely opening of the holes has no effect upon the unitary time evolution of the particle's wave packet other than to allow it to propagate as it would have in the complete absence of any walls. The rest of the analysis is the same as in Sec. 18.1, and shows that the conditional probabilities (18.8) also apply to the present situation: if the particle is later detected by C, it was in the c channel inside the box at t_1.

An alternative consistent family has for its support the single unitary history

$$[0a] \odot [1\bar{a}], \tag{18.26}$$

where $|1\bar{a}\rangle$ is the superposition state defined in (18.13). This family is clearly incompatible with the one in (18.25) because $[1\bar{a}]$ does not commute with either $[1c]$ or $[1d]$. Nonetheless, (18.26) is just as good a quantum description of the particle moving inside the closed box as is the pair of possibilities in (18.25). An experimental test which will confirm that the history (18.26) does, indeed, occur is shown in Fig. 18.3(c), and is only slightly more complicated than the one used earlier. Once again, holes are opened in the walls of the box just before the arrival of the particle, but now there are mirrors outside the holes and a second beam splitter, so one has a Mach-Zehnder interferometer. Let the path lengths be such that a particle in the state $|1\bar{a}\rangle$ at time t_1 will emerge from the second beam splitter in the f channel and trigger the detector F, whereas a particle in the orthogonal state

$$|1\bar{b}\rangle = (-|1c\rangle + |1d\rangle)/\sqrt{2} \tag{18.27}$$

will emerge in channel e and trigger E. The experiment needs to be repeated many times in order to get a statistically significant result, and if in every, or almost every run the particle is detected in F rather than E, one can infer that it was in the state $[1\bar{a}]$ at the earlier time t_1. That this is a plausible inference follows once again from the fact that quantum mechanics is a consistent theory abundantly confirmed by a variety of experimental tests.

It is obviously impossible to carry out the two types of measurements indicated in (b) and (c) of Fig. 18.3 on the same system during the same experimental run, and this is not surprising given the fact that while both (18.25) and (18.26) are valid quantum descriptions, they are mutually incompatible, so they cannot be applied to the same system at the same time. The "classical" macroscopic incompatibility of the two experimental arrangements, in the sense that setting up one of them prevents setting up the other, mirrors the quantum incompatibility of the microscopic events which are measured in the two cases. Thus an analysis using measurements can assist one in gaining an intuitive understanding of the incompatibility of quantum events and frameworks.

It has sometimes been suggested that certain conceptual difficulties associated with incompatible quantum frameworks could be resolved if there were a law of nature which specified the framework which had to be employed in any particular circumstance. That such an idea is not likely to work can be seen from the fact that either of the experiments indicated in Fig. 18.3 could in principle be carried out a large distance away and thus a long time after the particle emerges from the box, long enough to allow a choice to be made between the two experimental arrangements (see the discussion of delayed-choice in Ch. 20). Thus were there such a law of nature, it would either need to determine the choice of the later experiment, or allow that later choice to influence the particle while it was still inside the box. Neither of these seems very satisfactory.

A second example in which measurements are useful for understanding quantum incompatibility is shown in Fig. 18.4(a), in which a spin-half atom moving parallel to the y axis passes successively through two nondestructive Stern-Gerlach devices, represented schematically by squares, of the form shown in Fig. 18.2. At the times t_0, t_1, and t_2 the atom is (approximately) at the positions indicated by the dots in the figure. The first device measures S_z, and its unitary time development during the interval from t_0 to t_1 is given by (18.17). The second device measures S_x, and its unitary time development from t_1 to t_2 is given by

$$|x^+\rangle|X^\circ\rangle \mapsto |x^+\rangle|X^+\rangle, \quad |x^-\rangle|X^\circ\rangle \mapsto |x^-\rangle|X^-\rangle, \tag{18.28}$$

where $|X^\circ\rangle$, $|X^+\rangle$ and $|X^-\rangle$ are the initial state of the device and the states representing possible outcomes of the measurement.

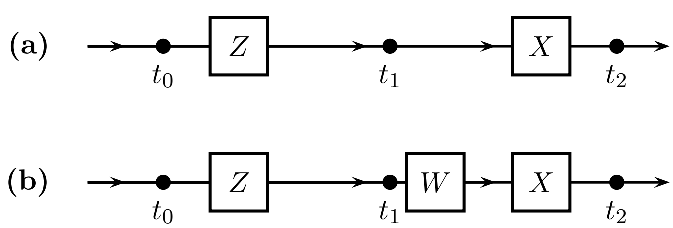

Figure 18.4: Spin-half atom passing through successive nondestructive Stern-Gerlach devices.

Given the starting state

$$|\Psi_0\rangle = |x^+\rangle|Z^\circ\rangle|X^\circ\rangle \tag{18.29}$$

at t_0, and that at t_2 the detectors are in the states Z^+ and X^+, what can one say about the spin of the atom at the time t_1 when it is midway between the two devices? A relatively coarse family whose support is the four histories

$$\Psi_0 \odot I \odot \{Z^+X^+,\, Z^+X^-,\, Z^-X^+,\, Z^-X^-\} \tag{18.30}$$

is useful for representing the initial data (see Sec. 16.1) of Ψ_0 at t_0 and Z^+X^+ at t_2.

The consistent family (18.30) can be refined in various ways. One possibility is to include information about S_z at t_1:

$$\Psi_0 \odot \begin{cases} [z^+] \odot \{Z^+X^+,\, Z^+X^-\}, \\ [z^-] \odot \{Z^-X^+,\, Z^-X^-\}. \end{cases} \tag{18.31}$$

Using this family one sees that

$$\Pr([z^+]_1 \,|\, Z_2^+ X_2^+) = 1, \tag{18.32}$$

so that the initial data imply that $S_z = +1/2$ at t_1. A different refinement includes information about S_x at t_1:

$$\Psi_0 \odot \begin{cases} [x^+] \odot \{Z^+X^+,\, Z^-X^+\} \\ [x^-] \odot \{Z^+X^-,\, Z^-X^-\}. \end{cases} \tag{18.33}$$

Using it one finds that

$$\Pr([x^+]_1 \,|\, Z_2^+ X_2^+) = 1, \tag{18.34}$$

so that in this framework the initial data imply that $S_x = +1/2$ at t_1. Since $[z^+]$ and $[x^+]$ do not commute, the frameworks (18.31) and (18.33) are incompatible, and the results (18.32) and (18.34) cannot be combined, even though each is correct in its own framework.

There is, of course, no experimental arrangement by means of which either (18.32) or (18.34) can be checked directly at the precise time t_1. The closest one can come is to insert another device W, as shown in Fig. 18.4(b), which carries out a nondestructive Stern-Gerlach measurement of S_w for some direction w at a time shortly after t_1. First consider the case $w = z$, so that the W apparatus repeats the measurement of the initial Z apparatus. One can show—the reader can easily work out the details—that with $w = z$, the Z and W devices have identical outcomes: Z^+W^+ or Z^-W^-. Thus if at t_2, when the atom has passed through all three devices, Z is in the state Z^+, W will be in the state W^+. This is precisely what one one would have anticipated on the basis of (18.32): the property $S_z = +1/2$ at t_1 was confirmed by the W measurement a short time later. In this sense the W device with $w = z$ confirms the correctness of a conclusion reached on the basis of the consistent family in (18.31). On the other hand, if $w = x$, so that the W apparatus measures S_x, a similar analysis shows that the X and W devices must have identical outcomes. In particular, if at t_2 X is in the state X^+, W will be in the state W^+, and this confirms the correctness of (18.34). Since the device W must have its field gradient (the gradient in the first magnet in Fig. 18.2) in a particular direction, it is obvious that in a particular experimental run either w is in the x or in the z direction, and cannot be in both directions simultaneously. The situation is thus similar to what we found in the previous example: a classical macroscopic incompatibility of the two measurement possibilities reflects the quantum incompatibility of the two frameworks (18.31) and (18.33).

How can we know that at time t_1 the atom had the property revealed a bit later by the spin measurement carried out by W? The answer to this question is the same as for its analog in the previous example. Quantum theory itself provides a consistent description of the situation, including the relevant connection between a property of the atom before a measurement takes place and the outcome of the measurement. One must, of course, employ an appropriate framework for this connection to be evident. For example, in the case $w = x$ one should use a consistent family with $[x^+]$ and $[x^-]$ at time t_1, for a family with $[z^+]$ and $[z^-]$ at t_1 cannot, obviously, be used to discuss the value of S_x.

There is, however, another concern which did not arise in the previous example using the beam splitter. The device W in Fig. 18.4(b) is located where it might conceivably disturb the later S_x measurement carried out by X. Can we say that the outcome of the latter, X^+ or X^-, is the same as it would have been, for this particular experimental run, had the apparatus W been absent, as in Fig. 18.4(a)? This is a *counterfactual* question: given a situation in which W is in fact present, it asks what *would* have happened *if*, contrary to fact, W had been absent. Answering counterfactual questions requires a further development, found in Sec. 19.4, of the principles of quantum reasoning discussed in Ch. 16. By using it one can argue that both for the case $w = x$ and also for the case $w = z$, had W been absent the X measurement outcome would have been the same.

18.5 General Nondestructive Measurements

In Sec. 17.5 we discussed a fairly general scheme for measurements, in general destructive, of the properties of a quantum system $\mathcal{S}$ corresponding to an orthonormal basis $\{|s^k\rangle\}$, by a measuring apparatus $\mathcal{M}$ initially in the state $|M_0\rangle$. To construct a corresponding description of nondestructive measurements, suppose that the unitary time development from t_0 to t_1 to t_2 corresponding to (17.30) is of the form

$$|s^k\rangle \otimes |M_0\rangle \mapsto |s^k\rangle \otimes |M_1\rangle \mapsto |s^k\rangle \otimes |M^k\rangle, \tag{18.35}$$

where the interaction between $\mathcal{S}$ and $\mathcal{M}$ takes place during the time interval from t_1 to t_2, and $\{|M^k\rangle\}$ is an orthonormal collection of states of $\mathcal{M}$ corresponding to the different measurement outcomes.

Given some initial state $|s_0\rangle$ which is a linear combination of the $\{|s^k\rangle\}$, (17.32), one can set up a consistent family analogous to (17.36) with support

$$\Psi_0 \odot \begin{cases} [s^1] \odot [s^1] \otimes M^1, \\ [s^2] \odot [s^2] \otimes M^2, \\ \dots\dots\dots\dots\dots \\ [s^n] \odot [s^n] \otimes M^n, \end{cases} \tag{18.36}$$

where $|\Psi_0\rangle$ is the state $|s_0\rangle|M_0\rangle$. Using this family one can show that M^k at t_2 implies s^k at t_1—(17.37) is valid with N^k replaced by M^k—and, in addition,

$$\Pr(s_2^j \,|\, M_2^k) = \delta_{jk} = \Pr(s_2^j \,|\, s_1^k). \tag{18.37}$$

The first equality tells us that if at t_2 the measurement outcome is M^k, the system $\mathcal{S}$ at this time is in the state $|s^k\rangle$, whereas the second shows that this measurement is nondestructive for the properties $\{[s^j]\}$.

Provided $\mathcal{S}$ and $\mathcal{M}$ do no interact for $t > t_2$, the later time development of $\mathcal{S}$ (e.g., what will happen if it interacts with a second measuring apparatus $\mathcal{M}'$) can be discussed using the method of conditional density matrices described in Sec. 15.7, with appropriate changes in notation: t_0, $\mathcal{A}$, and $\mathcal{B}$ of Sec. 15.7 become t_2, $\mathcal{S}$, and $\mathcal{M}$. Given a measurement outcome M^k, the corresponding conditional density matrix, see (15.61), is

$$\rho^k = [s^k], \tag{18.38}$$

and this can be used (typically as a pre-probability) as an initial state for the further time development of system $\mathcal{S}$. (If there is a second measuring apparatus $\mathcal{M}'$, one must, of course, also specify its initial state.)

One can also formulate a nondestructive counterpart to the measurement of a general decomposition of the identity $I_{\mathcal{S}} = \sum_k S^k$, (17.38), discussed in Sec. 17.5. Let the orthonormal basis $\{|s^{kl}\rangle\}$ be chosen so that $S^k = \sum_l [s^{kl}]$, (17.40), and assume a unitary time development

$$|s^{kl}\rangle \otimes |M_0\rangle \mapsto |s^{kl}\rangle \otimes |M_1\rangle \mapsto |s^{kl}\rangle \otimes |M^k\rangle, \tag{18.39}$$

where the apparatus state $|M^k\rangle$ corresponding to the $k'th$ outcome is assumed *not to depend upon* l. The counterpart of (17.43) is a consistent family with support

$$\Psi_0 \odot \begin{cases} S^1 \odot S^1 \otimes M^1, \\ S^2 \odot S^2 \otimes M^2, \\ \dots\dots\dots\dots\dots \\ S^n \odot S^n \otimes M^n, \end{cases} \tag{18.40}$$

and it yields conditional probabilities

$$\Pr(S_2^j \,|\, M_2^k) = \delta_{jk} = \Pr(S_2^j \,|\, S_1^k) \tag{18.41}$$

that are the obvious counterpart of (18.37). In addition, the outcome M^k at t_2 implies the property S^k at t_1: (17.45) holds with N^k replaced by M^k.

It is possible to refine (18.40) to give a more precise description at t_2. Define

$$|\sigma^k\rangle := S^k|s_0\rangle = \sum_l c_{kl}|s^{kl}\rangle, \tag{18.42}$$

using the expression (17.44) for $|s_0\rangle$. Then the unitary time development in (18.39) implies that

$$T(t_2, t_0)\big(|s_0\rangle \otimes |M_0\rangle\big) = \sum_k |\sigma^k\rangle \otimes |M^k\rangle. \tag{18.43}$$

As a consequence, the histories $\Psi_0 \odot S^k \odot (I - [\sigma^k]) \otimes M^k$ have zero weight, and

$$\Psi_0 \odot \begin{cases} S^1 \odot [\sigma^1] \otimes M^1, \\ S^2 \odot [\sigma^2] \otimes M^2, \\ \dots\dots\dots\dots\dots \\ S^n \odot [\sigma^n] \otimes M^n \end{cases} \tag{18.44}$$

is again the support of a consistent family. Indeed, one can produce an even finer family by replacing each S^k at t_1 with the corresponding $[\sigma^k]$.

In order to describe the later time development of $\mathcal{S}$, assuming no further interaction with $\mathcal{M}$ for $t > t_2$, one can again employ the method of conditional density matrices of Sec. 15.7, with

$$\rho^k = [\sigma^k] \tag{18.45}$$

at time t_2 corresponding to the measurement outcome M^k. If $\mathcal{S}$ is described by a density matrix ρ_0 at t_0, the corresponding result

$$\rho^k = S^k \rho_0 S^k / \mathrm{Tr}(S^k \rho_0 S^k) \tag{18.46}$$

is known as the *Lüders rule*. Note that the validity of both (18.41) and (18.42) depends on some fairly specific assumptions. If, for example, one were to suppose that

$$|s^{kl}\rangle \otimes |M_0\rangle \mapsto |s^{kl}\rangle \otimes |M_1\rangle \mapsto |s^{kl}\rangle \otimes |M^{kl}\rangle, \tag{18.47}$$

with the $\{|M^{kl}\rangle\}$ for different k and l an orthonormal collection, and define

$$M^k = \sum_l [M^{kl}] \tag{18.48}$$

as the projector corresponding to the k'th measurement outcome, (18.41) would still be valid, but neither (18.45) nor (18.46) would (in general) be correct.

The results in this section, like those in Sec. 17.5, can be generalized to the case of a macroscopic measuring apparatus using the approaches discussed in Sec. 17.4.