Chapter 11

Checking Consistency

11.1 Introduction

The conditions which define a consistent family of histories were stated in Ch. 10. The sample space must consist of a collection of mutually orthogonal projectors that add up to the history identity, and the chain operators for different members of the sample space must be mutually orthogonal, (10.20). Checking these conditions is in principle straightforward. In practice it can be rather tedious. Thus if there are n histories in the sample space, checking orthogonality involves computing n chain operators and then taking $n(n-1)/2$ operator inner products to check that they are mutually orthogonal. There are a number of simple observations, some definitions, and several "tricks" which can simplify the task of constructing a sample space of a consistent family, or checking that a given sample space is consistent. These form the subject matter of the present chapter. It is probably not worthwhile trying to read through this chapter as a unit. The reader will find it easier to learn the tricks by working through examples in Ch. 12 and later chapters, and referring back to this chapter as needed.

The discussion is limited to families in which all the histories in the sample space are of the product form, that is, represented by a projector on the history space which is a tensor product of quantum properties at different times, as in (8.7). As in the remainder of this book, the "strong" consistency conditions (10.20) are used rather than the weaker (10.25).

11.2 Support of a Consistent Family

A sample space of histories and the corresponding Boolean algebra it generates will be called *complete* if the sum of the projectors for the different histories in the sample space is the identity operator $\breve{I}$ on the history Hilbert space, (8.23). As noted at the end of Sec. 10.1, it is possible for the chain operator $K(Y)$ to be zero even if the history projector Y is not zero. The weight $W(Y) = \langle K(Y), K(Y)\rangle$ of such a history is obviously zero, so the history is dynamically impossible. Conversely, if $W(Y) = 0$, then $K(Y) = 0$; see the discussion in connection with (10.13). The *support* of a consistent family of histories is defined to be the set of all the histories in the sample space whose weight is strictly positive, that is, whose chain operators do not vanish. In other words, the support is what remains in the sample space if the histories of zero weight are removed. In general the support of a family is not complete, as that term was defined above, but one can say that it is *dynamically complete*.

When checking consistency, only histories lying in the support need be considered, because a chain operator which is zero is (trivially) orthogonal to all other chain operators. Using this fact can simplify the task of checking consistency in certain cases, such as the families considered in Ch. 12. Zero-weight histories are nonetheless of some importance, for they help to determine which histories, including histories of finite weight, are included in the Boolean event algebra. See the comments in Sec. 11.5 below.

11.3 Initial and Final Projectors

Checking consistency is often simplified by paying attention to the initial and final projectors of the histories in the sample space. Thus suppose that two histories

$$\begin{aligned} Y \, &= F_0 \odot F_1 \odot \cdots \odot F_f, \\ Y' &= F_0' \odot F_1' \odot \cdots \odot F_f' \end{aligned} \tag{11.1}$$

are defined for the same set of times $t_0 < t_1 < \cdots t_f$. If either $F_0 F_0' = 0$ or $F_f F_f' = 0$, then one can easily show, by writing out the corresponding trace and cycling operators around the trace, that $\langle K(Y), K(Y') \rangle = 0$. Consequently, one can sometimes tell by inspection that two chain operators will be orthogonal, without actually computing what they are.

If the sample space consists of histories with just two times $t_0 < t_1$, then the family is automatically consistent. The reason is that the product of the history projectors for two different histories in the sample space is zero (as the sample space consists of mutually exclusive possibilities). But in order that

$$(F_0 \odot F_1) \cdot (F_0' \odot F_1') = F_0 F_0' \odot F_1 F_1' \tag{11.2}$$

be zero, it is necessary that either $F_0 F_0'$ or $F_1 F_1'$ vanish. As we have just seen, either possibility implies that the chain operators for the two histories are orthogonal. As this holds for any pair of histories in the sample space, the consistency conditions are satisfied.

For families of histories involving three or more times, looking at the initial and final projectors does not settle the problem of consistency, but it does make checking consistency somewhat simpler. Suppose, for example, we are considering a family of histories based upon a fixed initial state Ψ_0 (see Sec. 11.5 below), with two possible projectors at the final time based upon the decomposition

$$t_f: \; I = P + \tilde{P}. \tag{11.3}$$

Then the sample space will consist of various histories, some of whose projectors will have P at the final time, and some $\tilde{P}$. The chain operator of a projector with a final P will be orthogonal to one with a final $\tilde{P}$. Thus we only need to check whether the chain operators for the histories ending in P are mutually orthogonal among themselves, and, similarly, the mutual orthogonality of the chain operators for histories ending in $\tilde{P}$. If the decomposition of the identity at the final time t_f involves more than two projectors, one need only check the orthogonality of chain operators for histories which end in the same projector, as it is automatic when the final projectors are different.

Yet another way of reducing the work involved in checking consistency can also be illustrated using (11.3). Suppose that at t_{f-1} there is a decomposition of the identity of the form

$$t_{f-1}: \ I = \sum_m Q_m, \tag{11.4}$$

and suppose that we have already checked that the chain operators for the different histories *ending in* P are all mutually orthogonal. In that case we can be sure that the chain operators for two histories with projectors

$$\begin{aligned} Y &= \Psi_0 \odot \cdots \odot Q_m \odot \tilde{P}, \\ Y' &= \Psi_0 \odot \cdots \odot Q_{m'} \odot \tilde{P} \end{aligned} \tag{11.5}$$

ending in $\tilde{P}$ will also be orthogonal to each other, provided $m' \neq m$. The reason is that by cycling operators around the trace in a suitable fashion one obtains an expression for the inner product of the chain operators in the form

$$\begin{aligned} \langle K(Y), K(Y')\rangle &= \mathrm{Tr}\left(\cdots Q_m T(t_{f-1}, t_f)\tilde{P}T(t_f, t_{f-1})Q_{m'}\right) \\ &= \mathrm{Tr}\left(\cdots Q_m Q_{m'}\right) - \mathrm{Tr}\left(\cdots Q_m T(t_{f-1}, t_f) P T(t_f, t_{f-1})Q_{m'}\right), \end{aligned} \tag{11.6}$$

where $\cdots$ refers to the same product of operators in each case. The second line of the equation is obtained from the first by replacing $\tilde{P}$ by $(I - P)$, using the linearity of the trace, and noting that $T(t_{f-1}, t_f)T(t_f, t_{f-1})$ is the identity operator, see (7.40). The trace of the product which contains $Q_m Q_{m'}$ vanishes, because $m' \neq m$ means that $Q_m Q_{m'} = 0$. The final trace vanishes because it is the inner product of the chain operators for the histories obtained from Y and Y' in (11.5) by replacing $\tilde{P}$ at the final position with P; by assumption, the orthogonality of these has already been checked. Thus the right side of (11.6) vanishes, so the chain operators for Y and Y' are orthogonal.

If the decomposition of the identity at t_f is into $n > 2$ projectors, the trick just discussed can still be used; however, it is necessary to check the mutual orthogonality of the chain operators for histories corresponding to each of $n - 1$ final projectors before one can obtain a certain number of results for those ending in the n'th projector "for free". If, rather than a fixed initial state Ψ_0, one is interested in a decomposition of the identity at t_0 involving several projectors, there is an analogous trick in which the projectors at t_1 play the role of the Q_m in the preceding discussion.

11.4 Heisenberg Representation

It is sometimes convenient to use the *Heisenberg representation* for the projectors and the chain operators, in place of the ordinary or *Schrödinger representation* which we have been using up to now. Suppose F_j is a projector representing an event thought of as happening at time t_j. We define the corresponding *Heisenberg projector* $\hat{F}_j$ using the formula

$$\hat{F}_j = T(t_r, t_j) F_j T(t_j, t_r), \tag{11.7}$$

where the *reference time* t_r is arbitrary, but must be kept fixed while analyzing a given family of histories. In particular, t_r cannot depend upon j. One can, for example, use $t_r = t_0$, but there are other possibilities as well. Given a history

$$Y = F_0 \odot F_1 \odot \cdots \odot F_f \tag{11.8}$$

of events at the times $t_0 < t_1 < \cdots t_f$, the *Heisenberg chain operator* is defined by:

$$\hat{K}(Y) = \hat{F}_f \hat{F}_{f-1} \cdots \hat{F}_0 = T(t_r, t_f) K(Y) T(t_0, t_r), \tag{11.9}$$

where the second equality is easily verified using the definition of $K(Y)$ in (10.5) along with (11.7). Note that $\hat{K}(Y)$, like $K(Y)$, is a linear function of its argument.

Now let Y' be a history similar to Y, except that each F_j in (11.8) is replaced by an event F'_j (which may or may not be the same as F_j). Then it is easy to show that

$$\langle K(Y'), K(Y)\rangle = \langle \hat{K}(Y'), \hat{K}(Y)\rangle = \mathrm{Tr}\left(\hat{F}'_0 \hat{F}'_1 \cdots \hat{F}'_f \hat{F}_f \hat{F}_{f-1} \cdots \hat{F}_0\right). \tag{11.10}$$

(Note that the inner product of the Heisenberg chain operators does not depend upon the choice of the reference time t_r.) Thus one obtains quite simple expressions for weights of histories and inner products of chain operators by using the Heisenberg representation. While this is not necessarily an advantage when doing an explicit calculation—time dependence has disappeared from (11.9), but one still has to use it to calculate the $\hat{F}_j$ in terms of the F_j, (11.7)—it does make some of the formulas simpler, and therefore more transparent. One disadvantage of using Heisenberg projectors is that, unlike ordinary (Schrödinger) projectors, they do not have a direct physical interpretation: what they signify in physical terms depends both on the form of the operator and on the dynamics of the quantum system.

11.5 Fixed Initial State

A family of histories for the times $t_0 < t_1 < \cdots < t_f$ based on an initial state Ψ_0 was introduced in Sec. 8.5, see (8.30). Let us write the elements of the sample space in the form

$$Y^\alpha = \Psi_0 \odot X^\alpha, \tag{11.11}$$

where for each α, X^α is a projector on the space $\bar{\mathcal{H}}$ of histories at times $t_1 < t_2 < \cdots < t_f$, with identity $\bar{I}$, and

$$\sum_\alpha X^\alpha = \bar{I}, \tag{11.12}$$

so that

$$\sum_\alpha Y^\alpha = \Psi_0 \odot \bar{I}. \tag{11.13}$$

The index α may have many components, as in the case of the product of sample spaces considered in Sec. 8.5. Since the Y^α do not add up to $\breve{I}$, we complete the sample space by adding an additional history

$$Z = (I - \Psi_0) \odot \bar{I}, \tag{11.14}$$

as in (8.31).

The chain operator $K(Z)$ is automatically orthogonal to the chain operators of all of the histories of the form (11.11) because the initial projectors are orthogonal, see Sec. 11.3 above. Consequently, the necessary and sufficient condition that the consistency conditions are satisfied for this sample space is that

$$\langle K(\Psi_0 \odot X^\alpha), K(\Psi_0 \odot X^\beta)\rangle = 0 \text{ for } \alpha \neq \beta. \tag{11.15}$$

As one normally assigns Ψ_0 probability 1 and $\tilde{\Psi}_0$ probability 0, the history Z can be ignored, and we shall henceforth assume that our sample space consists of the histories of the form (11.11).

One consequence of (11.12) and the fact that the chain operator $K(Y)$ is a linear function of Y, (10.7), is that

$$\sum_\alpha K(Y^\alpha) = K(\Psi_0 \odot \bar{I}) = T(t_f, t_0)\Psi_0. \tag{11.16}$$

Of course, this is still true if we omit all the zero terms from the sum on the left side, that is to say, if we sum only over histories in the support S of the sample space (as defined in Sec. 11.2):

$$\sum_{\alpha \in S} K(Y^\alpha) = K(\Psi_0 \odot \bar{I}) = T(t_f, t_0)\Psi_0. \tag{11.17}$$

One can sometimes make use of the result (11.17) in the following way. Suppose that we have found a certain collection S of histories of the form (11.11), represented by mutually orthogonal history projectors (i.e., $X^\alpha X^\beta = 0$ if $\alpha \neq \beta$) with non-zero weights. Suppose that, in addition, (11.17) is satisfied, but the X^α for $\alpha \in S$ do not add up to $\bar{I}$, (11.12). Can we be sure of finding a set of zero-weight histories of the form (11.11) so that we can complete our sample space in the sense that (11.13) is satisfied? Generally there are several ways of completing a sample space with histories of zero weight. One way is to define

$$X' = \bar{I} - \sum_{\alpha \in S} X^\alpha, \quad Y' = \Psi_0 \odot X'. \tag{11.18}$$

Then, since

$$Y' + \sum_{\alpha \in S} Y^\alpha = \Psi_0 \odot \bar{I}, \tag{11.19}$$

it follows from the linearity of K, see (11.17), that

$$K(Y') = 0. \tag{11.20}$$

Consequently, Y' as defined in (11.18) is a zero-weight history of the correct type, showing that there is at least one solution to our problem.

However, Y' might not be the sort of solution we are looking for. The point is that while zero-weight histories never occur, and thus in some sense they can be ignored, nonetheless they help to determine what constitutes the Boolean algebra of histories, since this depends upon the sample space. Sometimes one wants to discuss a particular item in the Boolean algebra which occurs with finite probability, but whose very presence in the algebra depends upon the existence of certain zero-weight histories in the sample space. In such a case one might need to use a collection of zero-weight history projectors adding up to Y' rather than Y' by itself.

The argument which begins at (11.16) looks a bit simpler if one uses the Heisenberg representation for the projectors and the chain operators. In particular, since

$$\hat{K}(\Psi_0 \odot \bar{I}) = \hat{\Psi}_0, \tag{11.21}$$

we can write (11.17) as

$$\sum_{\alpha \in S} \hat{K}(Y^\alpha) = \hat{\Psi}_0, \tag{11.22}$$

and since $\hat{K}(Y)$ is, like $K(Y)$, a linear function of its argument Y, the argument leading to $\hat{K}(Y') = 0$, obviously equivalent to $K(Y') = 0$, is somewhat more transparent.

11.6 Initial Pure State. Chain Kets

If the initial projector of Sec. 11.5 projects onto a pure state,

$$\Psi_0 = [\psi_0] = |\psi_0\rangle\langle\psi_0|, \tag{11.23}$$

where we will assume that $|\psi_0\rangle$ is normalized, there is an alternative route for calculating weights and checking consistency which involves using *chain kets* rather than chain operators. Since it is usually easier to manipulate kets than it is to carry out the corresponding tasks on operators, using chain kets has advantages in terms of both speed and simplicity. Suppose that Y^α in (11.11) has the form given in (8.30),

$$Y^\alpha = [\psi_0] \odot P_1^{\alpha_1} \odot P_2^{\alpha_2} \odot \cdots \odot P_f^{\alpha_f}, \tag{11.24}$$

with projectors at t_1, t_2, etc. drawn from decompositions of the identity of the type (8.25). Then it is easy to see that the corresponding chain operator is of the form

$$K(Y^\alpha) = |\alpha\rangle\langle\psi_0|, \tag{11.25}$$

where the *chain ket* $|\alpha\rangle$ is given by the expression

$$|\alpha\rangle = P_f^{\alpha_f} T(t_f, t_{f-1}) \cdots P_2^{\alpha_2} T(t_2, t_1) P_1^{\alpha_1} T(t_1, t_0)|\psi_0\rangle. \tag{11.26}$$

That is, start with $|\psi_0\rangle$, integrate Schrödinger's equation from t_0 to t_1, and apply the projector $P_1^{\alpha_1}$ to the result in order to obtain

$$|\phi_1\rangle = P_1^{\alpha_1} T(t_1, t_0)|\psi_0\rangle. \tag{11.27}$$

Next use $|\phi_1\rangle$ as the starting state, integrate Schrödinger's equation from t_1 to t_2, and apply $P_2^{\alpha_2}$. Continuing in this way will eventually yield $|\alpha\rangle$, where the symbol α stands for $(\alpha_1, \alpha_2, \ldots \alpha_f)$.

The inner product of two chain operators of the form (11.22) is the same as the inner product of the corresponding chain kets:

$$\begin{aligned}\langle K(Y^\alpha), K(Y^\beta)\rangle &= \mathrm{Tr}\Big(K^\dagger(Y^\alpha) K(Y^\beta)\Big) \\ &= \mathrm{Tr}\big(|\psi_0\rangle\langle\alpha|\beta\rangle\langle\psi_0|\big) = \langle\alpha|\beta\rangle.\end{aligned} \tag{11.28}$$

Consequently, the consistency condition becomes

$$\langle\alpha|\beta\rangle = 0 \text{ for } \alpha \neq \beta, \tag{11.29}$$

while the weight of a history is

$$W(Y^\alpha) = \langle\alpha|\alpha\rangle. \tag{11.30}$$

In the special case in which one of the projectors at time t_f projects onto a pure state $|\alpha_f\rangle$, the chain ket will be a complex constant, which could be zero, times $|\alpha_f\rangle$. If two or more histories in the sample space have the same final projector onto a pure state $|\alpha_f\rangle$, then consistency requires that at most one of these chain kets can be nonzero.

The analog of the argument in Sec. 11.5 following (11.17) leads to the following conclusion. Suppose one has a collection S of non-zero chain kets of the form (11.26) with the property that

$$\sum_{\alpha\in S} |\alpha\rangle = T(t_f, t_0)|\psi_0\rangle. \tag{11.31}$$

That is, they add up to the state produced by the unitary time evolution of $|\psi_0\rangle$ from t_0 to t_f. Suppose also that for the collection S the consistency conditions (11.29) are satisfied. Then one knows that the collection of histories $\{Y^\alpha : \alpha \in S\}$ is the support of a consistent family: there is at least one way (and usually there are many different ways) to add histories of zero weight to the support S in order to have a sample space satisfying (11.13), with $\Psi_0 = [\psi_0]$. Nonetheless, for the reasons discussed towards the end of Sec. 11.5, it is sometimes a good idea to go ahead and construct the zero-weight histories explicitly, in order to have a Boolean algebra of history projectors with certain specific properties, rather than relying on a general existence proof.

11.7 Unitary Extensions

For the following discussion it is convenient to use the Heisenberg representation introduced in Sec. 11.4, even though the concept of unitary extensions works equally well for the ordinary (Schrödinger) representation. Unitary histories were introduced in Sec. 8.7 and defined by (8.38). An equivalent definition is that the corresponding Heisenberg operators be identical,

$$\hat{F}_0 = \hat{F}_1 = \cdots = \hat{F}_f, \tag{11.32}$$

where we have used t_0 as the initial time rather than t_1 as in Sec. 8.7. It is obvious from (11.9) that the Heisenberg chain operator $\hat{K}$ for a unitary history is the projector $\hat{F}_0$.

Next suppose that in place of (11.32) we have

$$\hat{F}_0 = \hat{F}_1 = \cdots = \hat{F}_{m-1} \neq \hat{F}_m = \hat{F}_{m+1} = \cdots = \hat{F}_f, \tag{11.33}$$

where m is some integer in the interval $1 \leq m \leq f$. We shall call this a "one-jump history", because the Heisenberg projectors are not all equal; there is a change, or "jump" between t_{m-1} and t_m. In a one-jump history there are precisely two types of Heisenberg projectors, with all the projectors of one type occurring at times which are earlier than the first occurrence of a projector of the other type. The chain operator for a history with one jump is $\hat{K} = \hat{F}_f\hat{F}_0$. (If, as is usually the case, $\hat{F}_0$ and $\hat{F}_f$ do not commute, $\hat{K}$ is not a projector.) Similarly, a history with two jumps is characterized by

$$\hat{F}_0 = \cdots = \hat{F}_{m-1} \neq \hat{F}_m = \cdots = \hat{F}_{m'-1} \neq \hat{F}_{m'} = \cdots = \hat{F}_f, \tag{11.34}$$

with m and m' two integers in the range $1 \leq m < m' \leq f$, and its chain operator is $\hat{K} = \hat{F}_f\hat{F}_m\hat{F}_0$. (It could be the case that $\hat{F}_f = \hat{F}_0$.) Histories with three or more jumps are defined in a similar way.

A *unitary extension of a unitary history* (11.32) is obtained by adding some additional times, which may be earlier than t_0 or between t_0 and t_f or later than t_f; the only restriction is that the new times do not appear in the original list $t_0, t_1, \cdots t_f$. At each new time the projector for the event is chosen so that the corresponding Heisenberg projector is identical to those in the original history, (11.32). Hence, a unitary extension of a unitary history is itself a unitary history, and its Heisenberg chain operator is $\hat{F}_0$, the same as for the original history.

A *unitary extension of a history with one jump* is obtained by adding additional times, and requiring that the corresponding Heisenberg projectors are such that the new history has one jump. This means that if a new time t' precedes t_{m-1} in (11.33), the corresponding Heisenberg projector $\hat{F}'$ will be $\hat{F}_0$, whereas if it follows t_m, $\hat{F}'$ will be $\hat{F}_m$. If additional times are introduced between t_{m-1} and t_m, then the Heisenberg projectors corresponding to these times must all be $\hat{F}_0$, or all $\hat{F}_m$, or if some are $\hat{F}_0$ and some are $\hat{F}_m$, then all the times associated with the former must precede the earliest time associated with the latter. The Heisenberg chain operator of the extension is the same as for the original history, $\hat{F}_f \hat{F}_0$.

Unitary extensions of histories with two or more jumps follow the same pattern. One or more additional times are introduced, and the corresponding Heisenberg projectors must be such that the number of jumps in the new history is the same as in the original history. As a consequence, the Heisenberg chain operator is left unchanged. By using a limiting process it is possible to produce a unitary extension of a history in which events are defined on a continuous time interval. However, it is not clear that there is any advantage to doing so.

The fact that the Heisenberg chain operator is not altered in forming a unitary extension means that the weight W of an extended history is the same as that of the original history. Likewise, if the chain operators for a collection of histories are mutually orthogonal, the same is true for the chain operators of the unitary extensions. These results can be used to extend a consistent family of histories to include additional times without having to recheck the consistency conditions or recalculate the weights.

There is a slight complication in that while the histories obtained by unitary extension of the histories in the original sample space form the support of the new sample space, one needs additional zero-weight histories so that the projectors will add up to the history identity (or the projector for an initial state). The argument which follows shows that such zero-weight histories will always exist. Imagine that some history is being extended in steps, adding one additional time at each step. Suppose that t' has just been added to the set of times, with $\hat{F}'$ the corresponding Heisenberg projector. We now define a zero-weight history which has the same set of times as the newly extended history, and the same projectors at these times, except that at t' the projector $\hat{F}'$ is replaced with its complement

$$\hat{F}'' = I - \hat{F}'. \tag{11.35}$$

What is $\hat{K}''$ for the history containing $\hat{F}''$? Since the unitary extension had the same number of jumps as the original history, $\hat{F}''$ must occur next to an $\hat{F}'$ in the product which defines $\hat{K}''$, and this means that $\hat{K}'' = 0$, since $\hat{F}'\hat{F}'' = 0$. Thus we have produced a zero-weight history whose history projector when added to that of the newly extended history yields the projector for the history before the extension, since $\hat{F}' + \hat{F}'' = I$. Consequently, by carrying out unitary extensions in successive steps, at each step we generate zero-weight histories of the form needed to produce a final sample space in which all the history projectors add up to the desired answer. While the procedure just described can always be applied to generate a sample space, there will usually be other ways to add zero-weight histories, and since the choice of zero-weight histories can determine what events occur in the final Boolean algebra, as noted towards the end of Sec. 11.5, one may prefer to use some alternative to the procedure just described.

11.8 Intrinsically Inconsistent Histories

A single history is said to be *intrinsically inconsistent*, or simply *inconsistent*, if there is no consistent family which contains it as one of the elements of the Boolean algebra. The smallest Boolean algebra of histories which contains a history projector Y consists of 0, Y, $\tilde{Y} = \breve{I} - Y$, and the history identity $\breve{I}$. Since $Y\tilde{Y} = 0$,

$$\langle K(Y), K(\tilde{Y}) \rangle \neq 0, \tag{11.36}$$

see (10.21), is a necessary and sufficient condition that Y be intrinsically inconsistent.

If one restricts attention to histories which are product projectors, (8.6), no history involving just two times can be intrinsically inconsistent, so the simplest possibility is a three-time history of the form

$$Y = A \odot B \odot C. \tag{11.37}$$

Given Y, define the three histories

$$\begin{aligned} Y' &= A \odot \tilde{B} \odot C, \\ Y'' &= A \odot I \odot \tilde{C}, \\ Y''' &= \tilde{A} \odot I \odot I, \end{aligned} \tag{11.38}$$

where, as usual, $\tilde{P}$ stands for $I - P$. Then it is evident that

$$Y + Y' + Y'' + Y''' = I \odot I \odot I = \breve{I}, \tag{11.39}$$

so that

$$\tilde{Y} = Y' + Y'' + Y''', \tag{11.40}$$

and thus

$$K(\tilde{Y}) = K(Y') + K(Y'') + K(Y'''). \tag{11.41}$$

By considering initial and final projectors, Sec. 11.3, it is at once evident that $K(Y'')$ and $K(Y''')$ are orthogonal to $K(Y)$. Consequently,

$$\langle K(Y), K(\tilde{Y}) \rangle = \langle K(Y), K(Y') \rangle, \tag{11.42}$$

so that Y is an inconsistent history if the right side of this equation is non-zero.

As an example, consider the histories in (10.35), and let $Y = Y^1$. Then $Y' = Y^3$, and (10.37), which was used to show that (10.35) is an inconsistent family, also shows that Y^1 is intrinsically inconsistent. The same is true of Y^2, Y^3, and Y^4. The same basic strategy can be applied in certain cases which are at first sight more complicated; e.g., the histories in (13.19).