Chapter 10

Consistent Histories

10.1 Chain Operators and Weights

The previous chapter showed how the Born rule can be used to assign probabilities to a sample space of histories based upon an initial state $|\psi_0\rangle$ at t_0, and an orthonormal basis $\{|\phi_1^\alpha\rangle\}$ of the Hilbert space at a later time t_1. In this chapter we show how an extension of the Born rule can be used to assign probabilities to much more general families of histories, including histories defined at an arbitrary number of different times and using decompositions of the identity which are not limited to pure states, provided certain consistency conditions are satisfied.

We begin by rewriting the Born weight (9.22) for the history

$$Y^\alpha = [\psi_0] \odot [\phi_1^\alpha] \tag{10.1}$$

in the following way:

$$\begin{aligned} W(Y^\alpha) &= |\langle\phi_1^\alpha|T(t_1,t_0)|\psi_0\rangle|^2 = \langle\psi_0|T(t_0,t_1)|\phi_1^\alpha\rangle\langle\phi_1^\alpha|T(t_1,t_0)|\psi_0\rangle \\ &= \mathrm{Tr}\Big([\psi_0]\,T(t_0,t_1)\,[\phi_1^\alpha]\,T(t_1,t_0)\Big) = \mathrm{Tr}\Big(K^\dagger(Y^\alpha)K(Y^\alpha)\Big), \end{aligned} \tag{10.2}$$

where the *chain operator* $K(Y)$ and its adjoint are given by the expressions

$$K(F_0 \odot F_1) = F_1 T(t_1,t_0) F_0, \quad K^\dagger(F_0 \odot F_1) = F_0 T(t_0,t_1) F_1 \tag{10.3}$$

in the case of a history $Y = F_0 \odot F_1$ involving just two times; recall that $T(t_0,t_1) = T^\dagger(t_1,t_0)$. The steps from the left side to the right side of (10.2) are straightforward but not trivial, and the reader may wish to work through them. Recall that if $|\psi\rangle$ is any normalized ket, $[\psi] = |\psi\rangle\langle\psi|$ is the projector onto the one-dimensional subspace containing $|\psi\rangle$, and $\langle\psi|A|\psi\rangle$ is equal to $\mathrm{Tr}(|\psi\rangle\langle\psi|A) = \mathrm{Tr}([\psi]A)$.

For a general history of the form

$$Y = F_0 \odot F_1 \odot F_2 \odot \cdots \odot F_f \tag{10.4}$$

with events at times $t_0 < t_1 < t_2 < \cdots < t_f$ the chain operator is defined as

$$K(Y) = F_f T(t_f,t_{f-1}) F_{f-1} T(t_{f-1},t_{f-2}) \cdots T(t_1,t_0) F_0, \tag{10.5}$$

and its adjoint is given by the expression

$$K^\dagger(Y) = F_0 T(t_0, t_1) F_1 T(t_1, t_2) \cdots T(t_{f-1}, t_f) F_f. \tag{10.6}$$

Notice that the adjoint is formed by replacing each $\odot$ in (10.4) separating F_j from F_{j+1} by $T(t_j, t_{j+1})$. In both K and $K^\dagger$, each argument of any given T is adjacent to a projector representing an event at this particular time. One could just as well define $K(Y)$ using (10.6) and its adjoint $K^\dagger(Y)$ using (10.5). The definition used here is slightly more convenient for some purposes, but either convention yields exactly the same expressions for weights and consistency conditions, so there is no compelling reason to employ one rather than the other. Note that Y is an operator on the history Hilbert space $\breve{\mathcal{H}}$, and $K(Y)$ an operator on the original Hilbert space $\mathcal{H}$. Operators of these two types should not be confused with one another.

The definition of $K(Y)$ in (10.5) makes good sense when the F_j in (10.4) are any operators on the Hilbert space, not just projectors. In addition, K can be extended by linearity to sums of tensor product operators of the type (10.4):

$$K(Y' + Y'' + Y''' + \cdots) = K(Y') + K(Y'') + K(Y''') + \cdots . \tag{10.7}$$

In this way, K becomes a linear map of operators on the history space $\breve{\mathcal{H}}$ to operators on the Hilbert space $\mathcal{H}$ of a system at a single time, and it is sometimes useful to employ this extended definition.

The sequence of operators which make up the "chain" on the right side of (10.6) is in the same order as the sequence of times $t_0 < t_1 < \cdots t_f$. This is important; one does *not* get the same answer (in general) if the order is different. Thus for $f = 2$, with $t_0 < t_1 < t_2$, the operator defined by (10.6) is different from

$$L^\dagger(Y) = F_0 T(t_0, t_2) F_2 T(t_2, t_1) F_1, \tag{10.8}$$

and it is $K(Y)$, not $L(Y)$ which yields physically sensible results.

When all the projectors in a history are onto pure states, the chain operator has a particularly simple form when written in terms of dyads. For example, if

$$Y = |\psi_0\rangle\langle\psi_0| \odot |\phi_1\rangle\langle\phi_1| \odot |\omega_2\rangle\langle\omega_2|, \tag{10.9}$$

then the chain operator

$$K(Y) = \langle\omega_2|T(t_2, t_1)|\phi_1\rangle \cdot \langle\phi_1|T(t_1, t_0)|\psi_0\rangle \cdot |\omega_2\rangle\langle\psi_0| \tag{10.10}$$

is a product of complex numbers, often called *transition amplitudes*, with a dyad $|\omega_2\rangle\langle\psi_0|$ formed in an obvious way from the first and last projectors in the history.

Given any projector Y on the history space $\breve{\mathcal{H}}$, we assign to it a non-negative weight

$$W(Y) = \mathrm{Tr}[K^\dagger(Y) K(Y)] = \langle K(Y), K(Y)\rangle, \tag{10.11}$$

where the angular brackets on the right side denote an *operator inner product* whose general definition is

$$\langle A, B\rangle := \mathrm{Tr}[A^\dagger B], \tag{10.12}$$

with A and B any two operators on $\mathcal{H}$. In an infinite-dimensional space the formula (10.12) does not always make sense, since the trace of an operator is not defined if one cannot write it as a convergent sum. Technical issues can be avoided by restricting oneself to a finite-dimensional Hilbert space, where the trace is always defined, or to operators on infinite-dimensional spaces for which (10.12) does makes sense.

Operators on a Hilbert space $\mathcal{H}$ form a linear vector space under addition and multiplication by (complex) scalars. If $\mathcal{H}$ is n dimensional, its operators form an n^2-dimensional Hilbert space if one uses (10.12) to define the inner product. This inner product has all the usual properties: it is antilinear in its left argument, linear in its right argument, and satisfies:

$$\langle A, B\rangle^* = \langle B, A\rangle, \quad \langle A, A\rangle \geq 0, \tag{10.13}$$

with $\langle A, A\rangle = 0$ only if $A = 0$; see (3.92). Consequently, the weight $W(Y)$ defined by (10.11) is a non-negative real number, and it is zero if and only if the chain operator $K(Y)$ is zero. If one writes the operators as matrices using some fixed orthonormal basis of $\mathcal{H}$, one can think of them as n^2-component vectors, where each matrix element is one of the components of the vector. Addition of operators and multiplying an operator by a scalar then follow the same rules as for vectors, and the same is true of inner products. In particular, $\langle A, A\rangle$ is the sum of the absolute squares of the n^2 matrix elements of A.

If $\langle A, B\rangle = 0$, we shall say that the operators A and B are (mutually) *orthogonal*. Just as in the case of vectors in the Hilbert space, $\langle A, B\rangle = 0$ implies $\langle B, A\rangle = 0$, so orthogonality is a symmetrical relationship between A and B. Earlier we introduced a different definition of operator orthogonality by saying that two projectors P and Q are orthogonal if and only if $PQ = 0$. Fortunately, the new definition of orthogonality is an extension of the earlier one: if P and Q are projectors, they are also positive operators, and the argument following (3.93) in Sec. 3.9 shows that $\mathrm{Tr}[PQ] = 0$ if and only if $PQ = 0$.

It is possible to have a history with a non-vanishing projector Y for which $K(Y) = 0$. These histories (and only these histories) have zero weight. We shall say that they are *dynamically impossible*. They never occur, because they have probability zero. For example, $[z^+] \odot [z^-]$ for a spin half particle is dynamically impossible in the case of trivial dynamics, $T(t', t) = I$.

10.2 Consistency Conditions and Consistent Families

Classical weights of the sort used to assign probabilities in stochastic processes such as a random walk, see Sec. 9.1, have the property that they are *additive* functions on the sample space: if E and F are two disjoint collections of histories from the sample space, then, as in (9.11),

$$W(E \cup F) = W(E) + W(F). \tag{10.14}$$

If quantum weights are to function the same way as classical weights, they too must satisfy (10.14), or its quantum analog. Suppose that our sample space of histories is a decomposition $\{Y^\alpha\}$ of the history identity. Any projector Y in the corresponding Boolean algebra can be written as:

$$Y = \sum_\alpha \pi^\alpha Y^\alpha, \tag{10.15}$$

where each π^α is 0 or 1. Additivity of W then corresponds to:

$$W(Y) = \sum_\alpha \pi^\alpha W(Y^\alpha). \tag{10.16}$$

However, the weights defined using (10.11) do not, in general, satisfy (10.16). Since the chain operator is a linear map, (10.15) implies that:

$$K(Y) = \sum_\alpha \pi^\alpha K(Y^\alpha). \tag{10.17}$$

If we insert this in (10.11), and use the (anti)linearity of the operator inner product (note that the π^α are real), the result is

$$W(Y) = \sum_\alpha \sum_\beta \pi^\alpha \pi^\beta \langle K(Y^\alpha), K(Y^\beta)\rangle, \tag{10.18}$$

whereas the right side of (10.16) is given by:

$$\sum_\alpha \pi^\alpha W(Y^\alpha) = \sum_\alpha \pi^\alpha \langle K(Y^\alpha), K(Y^\alpha)\rangle. \tag{10.19}$$

In general, (10.18) and (10.19) will be different. However, in the case in which

$$\langle K(Y^\alpha), K(Y^\beta)\rangle = 0 \text{ for } \alpha \neq \beta, \tag{10.20}$$

only the diagonal terms $\alpha = \beta$ remain in the sum (10.18), so it is the same as (10.19), and the additivity condition (10.16) will be satisfied. Thus a sufficient condition for the quantum weights to be additive is that the chain operators associated with the different histories in the sample space be *mutually orthogonal* in terms of the inner product defined in (10.12). The approach we shall adopt is to limit ourselves to sample spaces of quantum histories for which the equalities in (10.20), known as *consistency conditions*, are fulfilled. Such sample spaces, or the corresponding Boolean algebras, will be referred to as *consistent families* of histories, or *frameworks*.

The consistency conditions in (10.20) are also called "decoherence conditions", and the terms "decoherent family", "consistent set" and "decoherent set" are sometimes used to denote a consistent family or framework. The adjective "consistent", as we have defined it, applies to families of histories, and not to individual histories. However, a single history Y can be said to be inconsistent if there is no consistent family which contains it as one of the members of its Boolean algebra. For an example see Sec. 11.8.

A consequence of the consistency conditions is the following. Let Y and $\bar{Y}$ be any two history projectors belonging to the Boolean algebra generated by the decomposition $\{Y^\alpha\}$. Then

$$Y\bar{Y} = 0 \text{ implies } \langle K(Y), K(\bar{Y})\rangle = 0. \tag{10.21}$$

To see that this is true, write Y and $\bar{Y}$ in the form (10.15), using coefficients $\bar{\pi}^\alpha$ for $\bar{Y}$. Then

$$Y\bar{Y} = \sum_\alpha \pi^\alpha \bar{\pi}^\alpha Y^\alpha, \tag{10.22}$$

so $Y\bar{Y} = 0$ implies that

$$\pi^\alpha \bar{\pi}^\alpha = 0 \tag{10.23}$$

for every α. Next use the expansion (10.17) for both $K(Y)$ and $K(\bar{Y})$, so that

$$\langle K(Y), K(\bar{Y})\rangle = \sum_{\alpha\beta} \pi^\alpha \bar{\pi}^\beta \langle K(Y^\alpha), K(Y^\beta)\rangle. \tag{10.24}$$

The consistency conditions (10.20) eliminate the terms with $\alpha \neq \beta$ from the sum, and (10.23) eliminates those with $\alpha = \beta$, so one arrives at (10.21). On the other hand, (10.21) implies (10.20) as a special case, since two different projectors in the decomposition $\{Y^\alpha\}$ are necessarily orthogonal to each other. Consequently, (10.20) and (10.21) are equivalent, and either one can serve as a definition of a consistent family.

While (10.20) is sufficient to ensure the additivity of W, (10.16), it is by no means necessary. It suffices to have

$$\mathrm{Re}[\langle K(Y^\alpha), K(Y^\beta)\rangle] = 0 \text{ for } \alpha \neq \beta, \tag{10.25}$$

where Re denotes the real part. We shall refer to these as "weak consistency conditions." Even weaker conditions may work in certain cases. The subject has not been exhaustively studied. However, the conditions in (10.20) are easier to apply in actual calculations than are any of the weaker conditions, and seem adequate to cover all situations of physical interest which have been studied up till now. Consequently, we shall refer to them from now on as "the consistency conditions", while leaving open the possibility that further study may indicate the need to use a weaker condition that enlarges the class of consistent families.

What about sample spaces for which the consistency condition (10.20) is *not* satisfied? What shall be our attitude towards *inconsistent* families of histories? Within the consistent history approach to quantum theory such families are "meaningless" in the sense that there is no way to assign them probabilities within the context of a stochastic time development governed by the laws of quantum dynamics. This is not the first time we have encountered something which is "meaningless" within a quantum formalism. In the usual Hilbert space formulation of quantum theory, it makes sense to describe a spin half particle as having its angular momentum along the $+z$ axis, or along the $+x$ axis, but trying to combine these two descriptions using "and" leads to something which lacks any meaning, because it does not correspond to any subspace in the quantum Hilbert space. See the discussion in Sec. 4.6. Consistency, on the other hand, is a more stringent condition, because a family of histories corresponding to an acceptable Boolean algebra of projectors on the history Hilbert space may still fail to satisfy the consistency conditions.

Consistency is always something which is *relative to dynamical laws*. As will be seen in an example in Sec. 10.3 below, changing the dynamics can render a consistent family inconsistent, or vice versa. Note that the conditions in (10.20) refer to an *isolated* quantum system. If a system is not isolated and is interacting with its environment, one must apply the consistency conditions to the system together with its environment, regarding the combination as an isolated system. A consistent family of histories for a system isolated from its environment may turn out to be inconsistent if interactions with the environment are "turned on." Conversely, interactions with the environment can sometimes ensure the consistency of a family of histories which would be inconsistent were the system isolated. Environmental effects go under the general heading of *decoherence*. (The term does not refer to the same thing as "decoherent" in "decoherent histories", though the two are related, and this sometimes causes confusion.) Decoherence is an active field of research, and while there has been considerable progress, there is much that is still not well understood. A brief introduction to the subject will be found in Ch. 26.

Must the orthogonality conditions in (10.20) be satisfied exactly, or should one allow small deviations from consistency? Inasmuch as the consistency conditions form part of the axiomatic structure of quantum theory, in the same sense as the Born rule discussed in the previous chapter, it is natural to require that they be satisfied exactly. On the other hand, as first pointed out by Dowker and Kent, it is plausible that when the off-diagonal terms $\langle K(Y^\alpha), K(Y^\beta)\rangle$ in (10.24) are small compared to the diagonal terms $\langle K(Y^\alpha), K(Y^\alpha)\rangle$, one can find a "nearby" family of histories in which the consistency conditions are satisfied exactly. A nearby family is one in which the original projectors used to define the events (properties at a particular time) making up the histories in the family are replaced by projectors onto nearby subspaces of the same dimension. For example, consider a projector $[\phi]$ onto the subspace spanned by a normalized ket $|\phi\rangle$. The subspace $[\chi]$ spanned by a second normalized ket $|\chi\rangle$ can be said to be near to $[\phi]$ provided $|\langle\chi|\phi\rangle|^2$ is close to 1; that is, if the angle ϵ defined by

$$\sin^2(\epsilon) = 1 - |\langle\chi|\phi\rangle|^2 = \tfrac{1}{2}\mathrm{Tr}\left[([\chi]-[\phi])^2\right] \tag{10.26}$$

is small. Notice that this measure is left unchanged by unitary time evolution: if $|\chi\rangle$ is near to $|\phi\rangle$ then $T(t',t)|\chi\rangle$ is near to $T(t',t)|\phi\rangle$. For example, if $|\phi\rangle$ corresponds to $S_z = +1/2$ for a spin-half particle, then a nearby $|\chi\rangle$ would correspond to $S_w = +1/2$ for a direction w close to the positive z axis. Or if $\phi(x)$ is a wave packet in one dimension, $\chi(x)$ might be the wave packet with its tails cut off, and then normalized. Of course, the histories in the nearby family are not the same as those in the original family. Nonetheless, since the subspaces which define the events are close to the original subspaces, their physical interpretation will be rather similar. In that case one would not commit a serious error by ignoring a small lack of consistency in the original family.

10.3 Examples of Consistent and Inconsistent Families

As a first example, consider the family of two-time histories

$$Y^k = [\psi_0]\odot[\phi_1^k], \quad Z = (I - [\psi_0])\odot I \tag{10.27}$$

used in Sec. 9.3 when discussing the Born rule. The chain operators

$$K(Y^k) = [\phi_1^k]T(t_1,t_0)[\psi_0] = \langle\phi_1^k|T(t_1,t_0)|\psi_0\rangle\cdot|\phi_1^k\rangle\langle\psi_0| \tag{10.28}$$

are mutually orthogonal because

$$\langle K(Y^k), K(Y^l)\rangle \propto \mathrm{Tr}\left(|\psi_0\rangle\langle\phi_1^k|\phi_1^l\rangle\langle\psi_0|\right) = \langle\psi_0|\psi_0\rangle\langle\phi_1^k|\phi_1^l\rangle \tag{10.29}$$

is zero for $k \neq l$. To complete the argument, note that

$$\langle K(Y^k), K(Z)\rangle = \mathrm{Tr}\left([\psi_0]T(t_0,t_1)[\phi_1^k]T(t_1,t_0)(I-[\psi_0])\right) \tag{10.30}$$

is zero, because one can cycle $[\psi_0]$ from the beginning to the end of the trace, and its product with $(I - [\psi_0])$ vanishes. Consequently, all the histories discussed in Ch. 9 are consistent, which justifies our having omitted any discussion of consistency when introducing the Born rule.

The same argument works if we consider a more general situation in which the initial state is a projector Ψ_0 onto a subspace which could have a dimension greater than 1, and instead of an orthonormal basis we consider a general decomposition of the identity in projectors

$$I = \sum_k P^k \tag{10.31}$$

at time t_1. The family of two-time histories

$$Y^k = \Psi_0 \odot P^k, \quad Z = (I - \Psi_0) \odot I \tag{10.32}$$

is again consistent, since for $k \neq l$

$$\langle K(Y^k), K(Y^l)\rangle \propto \mathrm{Tr}\left(\Psi_0 T(t_0, t_1) P^k P^l T(t_1, t_0)\Psi_0\right) = 0 \tag{10.33}$$

because $P^k P^l = 0$, while $\langle K(Y^k), Z\rangle = 0$ follows, as in (10.30), from cycling operators inside the trace. (This argument is a special case of the general result in Sec. 11.3 that any family based on just two times is automatically consistent.) The probability of Y^k is given by

$$\Pr(Y^k) = \mathrm{Tr}\left(P^k T(t_1, t_0)\Psi_0 T(t_0, t_1)\right) / \mathrm{Tr}\,(\Psi_0)\,, \tag{10.34}$$

which we shall refer to as the *generalized Born rule*. The factor of $1/\mathrm{Tr}(\Psi_0)$ is needed to normalize the probability when Ψ_0 projects onto a space of dimension greater than 1.

Another situation in which the consistency conditions are automatically satisfied is that of a unitary family as defined in Sec. 8.7. For a given initial state such a family contains one unitary history, (8.41), obtained by unitary time development of this initial state, and various nonunitary histories, such as (8.42). It is straightforward to show that the chain operator for any nonunitary history in such a family is zero, and that the chain operators for unitary histories with different initial states (belonging to the same decomposition of the identity) are orthogonal to one another. Thus the consistency conditions are satisfied. If the initial condition assigns probability 1 to a particular initial state, the corresponding unitary history occurs with probability 1, and zero probability is assigned to every other history in the family.

To find an example of an inconsistent family, one must look at histories defined at three or more times. Here is a fairly simple example for a spin-half particle. The five history projectors

$$\begin{aligned}
Y^0 &= [z^-] \odot \quad I \quad \odot \quad I,\\
Y^1 &= [z^+] \odot [x^+] \odot [z^+],\\
Y^2 &= [z^+] \odot [x^+] \odot [z^-],\\
Y^3 &= [z^+] \odot [x^-] \odot [z^+],\\
Y^4 &= [z^+] \odot [x^-] \odot [z^-]
\end{aligned} \tag{10.35}$$

defined at the three times $t_0 < t_1 < t_2$ form a decomposition of the history identity, and thus a sample space of histories. However, for trivial dynamics, $T = I$, the family is inconsistent. To show this it suffices to compute the chain operators using (10.10). In particular,

$$\begin{aligned}
K(Y^1) &= |\langle z^+|x^+\rangle|^2 \cdot |z^+\rangle\langle z^+|,\\
K(Y^3) &= |\langle z^+|x^-\rangle|^2 \cdot |z^+\rangle\langle z^+|
\end{aligned} \tag{10.36}$$

are not orthogonal, since $|\langle z^+|x^+\rangle|^2$ and $|\langle z^+|x^-\rangle|^2$ are both equal to 1/2; indeed,

$$\langle K(Y^1), K(Y^3)\rangle = 1/4. \tag{10.37}$$

Similarly, $K(Y^2)$ and $K(Y^4)$ are not orthogonal, whereas $K(Y^1)$ is orthogonal to $K(Y^2)$, and $K(Y^3)$ to $K(Y^4)$. In addition, $K(Y^0)$ is orthogonal to the chain operators of the other histories. Since consistency requires that all pairs of chain operators for distinct histories in the sample space be orthogonal, this is not a consistent family.

On the other hand, the same five histories in (10.35) can form a consistent family if one uses a suitable dynamics. Suppose that there is a magnetic field along the y axis, and the time intervals $t_1 - t_0$ and $t_2 - t_1$ are chosen in such a way that

$$T(t_1, t_0) = T(t_2, t_1) = R, \tag{10.38}$$

where R is the unitary operator such that

$$\begin{aligned} R|z^+\rangle &= |x^+\rangle, \quad R|z^-\rangle = |x^-\rangle, \\ R|x^+\rangle &= |z^-\rangle, \quad R|x^-\rangle = -|z^+\rangle, \end{aligned} \tag{10.39}$$

where the second line is a consequence of the first when one uses the definitions in (4.14). With this dynamics, Y^2 is a unitary history whose chain operator is orthogonal to that of Y^0, because of the orthogonal initial states, while the chain operators for Y^1, Y^3, and Y^4 vanish. Thus the consistency conditions are satisfied. That the family (10.35) is consistent for one choice of dynamics and inconsistent for another serves to emphasize the important fact, noted earlier, that consistency depends upon the dynamical law of time evolution. This is not surprising given that the probabilities assigned to histories depend upon the dynamical law.

A number of additional examples of consistent and inconsistent histories will be discussed in Chs. 12 and 13. However, checking consistency by the process of finding chain operators for every history in a sample space is rather tedious and inefficient. Some general principles and various tricks explained in the next chapter make this process a lot easier. However, the reader may prefer to move on to the examples, and only refer back to Ch. 11 as needed.

10.4 Refinement and Compatibility

The refinement of a sample space of histories was discussed in Sec. 8.6. In essence, the idea is the same as for any other quantum sample space: some or perhaps all of the projectors in a decomposition of the identity are replaced by two or more finer projectors whose sum is the coarser projector. It is important to note that even if the coarser family one starts with is consistent, the finer family need not be consistent.

Suppose that $\mathcal{Z} = \{Z^\beta\}$ is a consistent sample space of histories, $\mathcal{Y} = \{Y^\alpha\}$ is a refinement of $\mathcal{Z}$, and that

$$Z^1 = Y^1 + Y^2. \tag{10.40}$$

Then, by linearity,

$$K(Z^1) = K(Y^1) + K(Y^2). \tag{10.41}$$

When a vector is written as a sum of two other vectors, the latter need not be perpendicular to each other, and by analogy, there is no reason to suppose that the terms on the right side of (10.3) are mutually orthogonal, $\langle K(Y^1), K(Y^2)\rangle = 0$, as is necessary if $\mathcal{Y}$ is to be a consistent family. Another way in which $\mathcal{Y}$ may fail to be consistent is the following. Since $\mathcal{Y}$ is a refinement of $\mathcal{Z}$, any projector in the sample space $\mathcal{Z}$, for example Z^3, belongs to the Boolean algebra generated by $\mathcal{Y}$. Because they represent mutually exclusive events, $Z^3 Z^1 = 0$, and because Y^1 and Y^2 in (10.40) are projectors, this means that

$$Z^3 Y^1 = 0 = Z^3 Y^2. \tag{10.42}$$

In addition, the consistency of $\mathcal{Z}$ implies that

$$\langle K(Z^3), K(Z^1)\rangle = \langle K(Z^3), K(Y^1)\rangle + \langle K(Z^3), K(Y^2)\rangle = 0. \tag{10.43}$$

However, this does not mean that either $\langle K(Z^3), K(Y^1)\rangle$ or $\langle K(Z^3), K(Y^2)\rangle$ is zero, whereas (10.21) implies, given (10.42), that both must vanish in order for $\mathcal{Y}$ to be consistent.

An example which illustrates these principles is the family $\mathcal{Y}$ whose sample space is (10.35), regarded as a refinement of the coarser family $\mathcal{Z}$ whose sample space consists of the three projectors Y^0, $Y^1 + Y^2$, and $Y^3 + Y^4$. As the histories in $\mathcal{Z}$ depend (effectively) on only two times, t_0 and t_1, the consistency of this family is a consequence of the general argument for the first example in Sec. 10.3. However, the family $\mathcal{Y}$ is inconsistent for $T(t', t) = I$.

In light of these considerations, we shall say that two or more consistent families are *compatible* if and only if they have a common refinement which is itself a *consistent* family. In order for two consistent families $\mathcal{Y}$ and $\mathcal{Z}$ to be compatible, two conditions, taken together, are necessary and sufficient. First, the projectors for the two sample spaces, or decompositions of the history identity, $\{Y^\alpha\}$ and $\{Z^\beta\}$ must commute with each other:

$$Y^\alpha Z^\beta = Z^\beta Y^\alpha \text{ for all } \alpha, \beta. \tag{10.44}$$

Second, the chain operators associated with distinct projectors of the form $Y^\alpha Z^\beta$ must be mutually orthogonal:

$$\langle K(Y^\alpha Z^\beta), K(Y^{\hat\alpha} Z^{\hat\beta})\rangle = 0 \text{ if } \alpha \neq \hat\alpha \text{ or } \beta \neq \hat\beta. \tag{10.45}$$

Note that (10.45) is automatically satisfied when $Y^\alpha Z^\beta = 0$, so one only needs to check this condition for non-zero products. Similar considerations apply in an obvious way to three or more families. Consistent families that are not compatible are said to be (mutually) *incompatible*.