Chapter 4

Physical Properties

4.1 Classical and Quantum Properties

We shall use the term *physical property* to refer to something which can be said to be either *true* or *false* for a particular physical system. Thus "the energy is between 10 and 12 μJ" or "the particle is between x_1 and x_2" are examples of physical properties. One must distinguish between a *physical property* and a *physical variable*, such as the position or energy or momentum of a particle. A physical variable can take on different numerical values, depending upon the state of the system, whereas a physical property is either a true or a false description of a particular physical system at a particular time. A physical variable taking on a particular value, or lying in some range of values, is an example of a physical property.

In the case of a classical mechanical system, a physical property is always associated with some subset of points in its phase space. Consider, for example, a harmonic oscillator whose phase space is the x, p plane shown in Fig. 2.1 on page 9. The property that its energy is equal to some value $E_0 > 0$ is associated with a set of points on an ellipse centered at the origin. The property that the energy is less than E_0 is associated with the set of points inside this ellipse. The property that the position x lies between x_1 and x_2 corresponds to the set of points inside a vertical band shown crosshatched in this figure, and so forth.

Given a property P associated with a set of points $\mathcal{P}$ in the phase space, it is convenient to introduce an *indicator function*, or *indicator* for short, $P(\gamma)$, where γ is a point in the phase space, defined in the following way:

$$P(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

(It is convenient to use the same symbol P for a property and for its indicator, as this will cause no confusion.) Thus if at some instant of time the phase point γ_0 associated with a particular physical system is inside the set $\mathcal{P}$, then $P(\gamma_0) = 1$, meaning that the system possesses this property, or the property is true. Similarly, if $P(\gamma_0) = 0$, the system does not possess this property, so for this system the property is false.

A *physical property of a quantum system* is associated with a *subspace* $\mathcal{P}$ of the quantum Hilbert space $\mathcal{H}$ in much the same way as a physical property of a classical system is associated with a subset of points in its phase space, and the *projector* P onto $\mathcal{P}$, Sec. 3.4, plays a role analogous to the classical indicator function. If the quantum system is described by a ket $|\psi\rangle$ which lies in the subspace $\mathcal{P}$, so that $|\psi\rangle$ is an eigenstate of P with eigenvalue 1,

$$P|\psi\rangle = |\psi\rangle, \tag{4.2}$$

one can say that the quantum system has the property P. On the other hand, if $|\psi\rangle$ is an eigenstate of P with eigenvalue 0,

$$P|\psi\rangle = 0, \tag{4.3}$$

then the quantum system does not have the property P, or, equivalently, it has the property $\tilde{P}$ which is the negation of P, see Sec. 4.4 below. When $|\psi\rangle$ is *not* an eigenstate of P, a situation with no analog in classical mechanics, we shall say that the property P is *undefined* for the quantum system.

4.2 Toy Model and Spin Half

In this section we will consider various physical properties associated with a toy model and with a spin-half particle, and in Sec. 4.3 properties of a continuous quantum system, such as a particle with a wave function $\psi(x)$. In Sec. 2.5 we introduced a toy model with wave function $\psi(m)$, where the position variable m is an integer restricted to taking on one of the $M = M_a + M_b + 1$ values in the range

$$-M_a \leq m \leq M_b. \tag{4.4}$$

In (2.26) we defined a wave function $\chi_n(m) = \delta_{mn}$ whose physical significance is that the particle is at the site (or in the cell) n. Let $|n\rangle$ be the corresponding Dirac ket. Then (2.24) tells us that

$$\langle k|n\rangle = \delta_{kn}, \tag{4.5}$$

so the kets $\{|n\rangle\}$ form an orthonormal basis of the Hilbert space.

Any scalar multiple $\alpha|n\rangle$ of $|n\rangle$, where α is a nonzero complex number, has precisely the same physical significance as $|n\rangle$. The set of all kets of this form together with the zero ket, i.e., the set of all multiples of $|n\rangle$, form a one-dimensional subspace of $\mathcal{H}$, and the projector onto this subspace, see Sec. 3.4, is

$$[n] = |n\rangle\langle n|. \tag{4.6}$$

Thus it is natural to associate the property that the particle is at the site n (something which can be true or false) with this subspace or, equivalently, with the corresponding projector, since there is a one-to-one correspondence between subspaces and projectors.

Since the projectors [0] and [1] for sites 0 and 1 are orthogonal to each other, their sum is also a projector

$$R = [0] + [1]. \tag{4.7}$$

The subspace $\mathcal{R}$ onto which R projects is two-dimensional and consists of all linear combinations of $|0\rangle$ and $|1\rangle$, that is, all states of the form

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{4.8}$$

Equivalently, it corresponds to all wave functions $\psi(m)$ which vanish when m is unequal to 0 or 1. The physical significance of $\mathcal{R}$, see the discussion in Sec. 2.3, is that the toy particle is *not outside* the interval $[0,1]$, where, since we are using a discrete model, the interval $[0,1]$ consists of the two sites $m=0$ and $m=1$. One can interpret "not outside" as meaning "inside", provided that is not understood to mean "at one or the other of the two sites $m=0$ or $m=1$."

The reason one needs to be cautious is that a typical state in $\mathcal{R}$ will be of the form (4.8) with both α and β unequal to zero. Such a state does not have the property that it is at $m=0$, for all states with this property are scalar multiples of $|0\rangle$, and $|\phi\rangle$ is not of this form. Indeed, $|\phi\rangle$ is not an eigenstate of the projector $[0]$ representing the property $m=0$, and hence according to the definition given at the end of Sec. 4.1, the property $m=0$ is undefined. The same comments apply to the property $m=1$. Thus it is certainly incorrect to say that the particle is either at 0 or at 1. Instead, the particle is represented by a delocalized wave, as discussed in Sec. 2.3. There are some states in $\mathcal{R}$ which are localized at 0 or localized at 1, but since $\mathcal{R}$ also contains other, delocalized, states, the property corresponding to $\mathcal{R}$ or its projector R, which holds for *all* states in this subspace, needs to be expressed by some English phrase other than "at 0 or 1". The phrases "not outside the interval $[0,1]$" or "no place other than 0 or 1," while they are a bit awkward, come closer to saying what one wants to say. The way to be perfectly precise is to use the projector R itself, since it is a precisely defined mathematical quantity. But of course one needs to build up an intuitive picture of what it is that R means.

Part of the process of building up one's intuition about the meaning of R comes from noting that (4.7) is not the only way of writing it as a sum of two orthogonal projectors. Another possibility is

$$R = [\sigma] + [\tau], \tag{4.9}$$

where

$$|\sigma\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}, \quad |\tau\rangle = (|0\rangle - i|1\rangle)/\sqrt{2} \tag{4.10}$$

are two normalized states in $\mathcal{R}$ which are mutually orthogonal. To check that (4.9) is correct, one can work out the dyad

$$\begin{aligned} |\sigma\rangle\langle\sigma| &= \tfrac{1}{2}\big(|0\rangle + i|1\rangle\big)\big(\langle 0| - i\langle 1|\big) \\ &= \tfrac{1}{2}\big(|0\rangle\langle 0| + |1\rangle\langle 1| + i|1\rangle\langle 0| - i|0\rangle\langle 1|\big), \end{aligned} \tag{4.11}$$

where $\langle\sigma| = (|\sigma\rangle)^\dagger$ has been formed using the rules for the dagger operation (note the complex conjugate) in (3.12), and (4.11) shows how one can conveniently multiply things out to find the resulting projector. The dyad $|\tau\rangle\langle\tau|$ is the same except for the signs of the imaginary terms, so adding this to $|\sigma\rangle\langle\sigma|$ gives R. There are many other ways besides (4.9) to write R as a sum of two orthogonal projectors. In fact, given *any* normalized state in $\mathcal{R}$, one can always find another normalized state orthogonal to it, and the sum of the dyads corresponding to these two states is equal to R. The fact that R can be written as a sum $P+Q$ of two orthogonal projectors P and Q in many different ways is one reason to be cautious in assigning R a physical interpretation of "property P or property Q", although there are occasions, as we shall see later, when such an interpretation is appropriate.

The simplest non-trivial toy model has only $M=2$ sites, and it is convenient to discuss it using language appropriate to the spin degree of freedom of a particle of spin 1/2. Its Hilbert space $\mathcal{H}$

consists of all linear combinations of two mutually orthogonal and normalized states which will be denoted by $|z^+\rangle$ and $|z^-\rangle$, and which one can think of as the counterparts of $|0\rangle$ and $|1\rangle$ in the toy model. (In the literature they are often denoted by $|+\rangle$ and $|-\rangle$, or by $|\uparrow\rangle$ and $|\downarrow\rangle$.) The normalization and orthogonality conditions are

$$\langle z^+|z^+\rangle = 1 = \langle z^-|z^-\rangle, \quad \langle z^+|z^-\rangle = 0. \tag{4.12}$$

The physical significance of $|z^+\rangle$ is that the z component S_z of the internal or "spin" angular momentum has a value of $+1/2$ in units of $\hbar$, while $|z^-\rangle$ means that $S_z = -1/2$ in the same units. One sometimes refers to $|z^+\rangle$ and $|z^-\rangle$ as "spin up" and "spin down" states.

The two-dimensional Hilbert space $\mathcal{H}$ consists of all linear combinations of the form

$$\alpha|z^+\rangle + \beta|z^-\rangle, \tag{4.13}$$

where α and β are any complex numbers. It is convenient to parameterize these states in the following way. Let w denote a direction in space corresponding to ϑ, φ in spherical polar coordinates. For example, $\vartheta = 0$ is the $+z$ direction, while $\vartheta = \pi/2$ and $\varphi = \pi$ is the $-x$ direction. Then define the two states

$$\begin{aligned} |w^+\rangle &= +\cos(\vartheta/2)e^{-i\varphi/2}|z^+\rangle + \sin(\vartheta/2)e^{i\varphi/2}|z^-\rangle, \\ |w^-\rangle &= -\sin(\vartheta/2)e^{-i\varphi/2}|z^+\rangle + \cos(\vartheta/2)e^{i\varphi/2}|z^-\rangle. \end{aligned} \tag{4.14}$$

These are normalized and mutually orthogonal,

$$\langle w^+|w^+\rangle = 1 = \langle w^-|w^-\rangle, \quad \langle w^+|w^-\rangle = 0, \tag{4.15}$$

as a consequence of (4.12).

The physical significance of $|w^+\rangle$ is that S_w, the component of spin angular momentum in the w direction, has a value of $1/2$, whereas for $|w^-\rangle$, S_w has the value $-1/2$. For $\vartheta = 0$, $|w^+\rangle$ and $|w^-\rangle$ are the same as $|z^+\rangle$ and $|z^-\rangle$, respectively, apart from a phase factor, $e^{-i\varphi}$, which does not change their physical significance. For $\vartheta = \pi$, $|w^+\rangle$ and $|w^-\rangle$ are the same as $|z^-\rangle$ and $|z^+\rangle$, respectively, apart from phase factors. Suppose that w is a direction which is neither along nor opposite to the z axis, for example, $w = x$. Then both α and β in (4.13) are non-zero, and $|w^+\rangle$ does not have the property $S_z = +1/2$, nor does it have the property $S_z = -1/2$. The same is true if S_z is replaced by S_v, where v is any direction which is not the same as w or opposite to w. The situation is analogous to that discussed earlier for the toy model: think of $|z^+\rangle$ and $|z^-\rangle$ as corresponding to the states $|m = 0\rangle$ and $|m = 1\rangle$.

Any non-zero wave function (4.13) can be written as a complex number times $|w^+\rangle$ for a suitable choice of the direction w. For $\beta = 0$, the choice $w = z$ is obvious, while for $\alpha = 0$ it is $w = -z$. For other cases, write (4.13) in the form

$$\beta\left[(\alpha/\beta)|z^+\rangle + |z^-\rangle\right]. \tag{4.16}$$

A comparison with the expression for $|w^+\rangle$ in (4.14) shows that ϑ and φ are determined by the equation

$$e^{-i\varphi}\cot(\vartheta/2) = \alpha/\beta, \tag{4.17}$$

which, since α/β is finite (neither 0 nor ∞) always has a unique solution in the range

$$0 \le \varphi < 2\pi, \quad 0 < \vartheta < \pi. \tag{4.18}$$

4.3 Continuous Quantum Systems

This section deals with the quantum properties of a particle in one dimension described by a wave function $\psi(x)$ depending on the continuous variable x. Similar considerations apply to a particle in three dimensions with a wave function $\psi(\boldsymbol{r})$, and the same general approach can be extended to apply to collections of several particles. Quantum properties are again associated with subspaces of the Hilbert space $\mathcal{H}$, and since $\mathcal{H}$ is infinite-dimensional, these subspaces can be either finite or infinite-dimensional; we shall consider examples of both. (For infinite-dimensional subspaces one adds the technical requirement that they be closed, as defined in books on functional analysis.)

As a first example, consider the property that a particle lies inside (which is to say, not outside) the interval

$$x_1 \leq x \leq x_2, \tag{4.19}$$

with $x_1 < x_2$. As pointed out in Sec. 2.3, the (infinite-dimensional) subspace $\mathcal{X}$ which corresponds to this property consists of all wave functions which vanish for x outside the interval (4.19). The projector X associated with $\mathcal{X}$ is defined as follows. Acting on some wave function $\psi(x)$, X produces a new function $\psi_X(x)$ which is identical to $\psi(x)$ for x inside the interval (4.19), and zero outside this interval:

$$\psi_X(x) = X\psi(x) = \begin{cases} \psi(x) & \text{for } x_1 \leq x \leq x_2, \\ 0 & \text{for } x < x_1 \text{ or } x > x_2. \end{cases} \tag{4.20}$$

Note that X leaves unchanged any function which belongs to the subspace $\mathcal{X}$, so it acts as the identity operator on this subspace. If a wave function $\omega(x)$ vanishes throughout the interval (4.19), it will be orthogonal to all the functions in $\mathcal{X}$, and X applied to $\omega(x)$ will yield a function which is everywhere equal to zero. Thus X has the properties one would expect for a projector as discussed in Sec. 3.4.

One can write (4.20) using Dirac notation as

$$|\psi_X\rangle = X|\psi\rangle, \tag{4.21}$$

where the element $|\psi\rangle$ of the Hilbert space can be represented either as a position wave function $\psi(x)$ or as a momentum wave function $\hat{\psi}(p)$, the Fourier transform of $\psi(x)$, see (2.15). The relationship (4.21) can also be expressed in terms of momentum wave functions as

$$\hat{\psi}_X(p) = \int \hat{\xi}(p - p')\hat{\psi}(p')\, dp', \tag{4.22}$$

where $\hat{\psi}_X(p)$ is the Fourier transform of $\psi_X(x)$, and $\hat{\xi}(p)$ the Fourier transform of

$$\xi(x) = \begin{cases} 1 & \text{for } x_1 \leq x \leq x_2, \\ 0 & \text{for } x < x_1 \text{ or } x > x_2. \end{cases} \tag{4.23}$$

Whereas (4.20) is the most straightforward way to define $X|\psi\rangle$, it is important to note that the expression (4.22) is completely equivalent.

As another example, consider the property that the momentum of a particle lies in (i.e., not outside) the interval

$$p_1 \leq p \leq p_2. \tag{4.24}$$

This property corresponds, Sec. 2.4, to the subspace $\mathcal{P}$ of momentum wave functions $\hat{\psi}(p)$ which vanish outside this interval. The projector P corresponding to $\mathcal{P}$ can be defined by

$$\hat{\psi}_P(p) = P\hat{\psi}(p) = \begin{cases} \hat{\psi}(p) & \text{for } p_1 \leq p \leq p_2, \\ 0 & \text{for } p < p_1 \text{ or } p > p_2, \end{cases} \tag{4.25}$$

and in Dirac notation (4.25) takes the form

$$|\psi_P\rangle = P|\psi\rangle. \tag{4.26}$$

One could also express the position wave function $\psi_P(x)$ in terms of $\psi(x)$ using a convolution integral analogous to (4.22).

As a third example consider a one-dimensional harmonic oscillator. In textbooks it is shown that the energy E of an oscillator with angular frequency ω takes on a discrete set of values

$$E = n + \tfrac{1}{2},\ n = 0, 1, 2, \ldots \tag{4.27}$$

in units of $\hbar\omega$. Let $\phi_n(x)$ be the normalized wave function for energy $E = n + 1/2$, and $|\phi_n\rangle$ the corresponding ket. The one-dimensional subspace of $\mathcal{H}$ consisting of all scalar multiples of $|\phi_n\rangle$ represents the property that the energy is $n + 1/2$. The corresponding projector is the dyad $[\phi_n]$. When this projector acts on some $|\psi\rangle$ in $\mathcal{H}$, the result is

$$|\bar{\psi}\rangle = [\phi_n]|\psi\rangle = |\phi_n\rangle\langle\phi_n|\psi\rangle = \langle\phi_n|\psi\rangle\,|\phi_n\rangle, \tag{4.28}$$

that is, $|\phi_n\rangle$ multiplied by the scalar $\langle\phi_n|\psi\rangle$. One can write (4.28) using wave functions in the form

$$\bar{\psi}(x) = \int P(x, x')\psi(x')\,dx', \tag{4.29}$$

where

$$P(x, x') = \phi_n(x)\phi_n^*(x') \tag{4.30}$$

corresponds to the dyad $|\phi_n\rangle\langle\phi_n|$.

Since the states $|\phi_n\rangle$ for different n are mutually orthogonal, the same is true of the corresponding projectors $[\phi_n]$. Using this fact makes it easy to write down projectors for the energy to lie in some interval which includes two or more energy eigenvalues. For example, the projector

$$Q = [\phi_1] + [\phi_2] \tag{4.31}$$

onto the two-dimensional subspace of $\mathcal{H}$ consisting of linear combinations of $|\phi_1\rangle$ and $|\phi_2\rangle$ expresses the property that the energy E (in units of $\hbar\omega$) lies inside some interval such as

$$1 < E < 3, \tag{4.32}$$

where the choice of endpoints of the interval is somewhat arbitrary, given that the energy is quantized and takes on only discrete values; any other interval which includes 1.5 and 2.5, but excludes 0.5 and 3.5, would be just as good. The action of Q on a wave function $\psi(x)$ can be written as

$$\bar{\psi}(x) = Q\psi(x) = \int Q(x, x')\psi(x')\,dx', \tag{4.33}$$

with

$$Q(x, x') = \phi_1(x)\phi_1^*(x') + \phi_2(x)\phi_2^*(x'). \tag{4.34}$$

Once again, it is important not to interpret "energy lying inside the interval (4.32)" as meaning that it either has the value 1.5 or that it has the value 2.5. The subspace onto which Q projects also contains states such as $|\phi_1\rangle + |\phi_2\rangle$, for which the energy cannot be defined more precisely than by saying that it does not lie outside the interval, and thus the physical property expressed by Q cannot have a meaning which is more precise than this.

4.4 Negation of Properties (NOT)

A physical property can be true or false in the sense that the statement that a particular physical system at a particular time possesses a physical property can be either true or false. Books on logic present simple *logical operations* by which statements which are true or false can be transformed into other statements which are true or false. We shall consider three operations which can be applied to physical properties: negation, taken up in this section, and conjunction and disjunction, taken up in Sec. 4.5. In addition, quantum properties are sometimes incompatible or "non comparable", a topic discussed in Sec. 4.6.

As noted in Sec. 4.1, a classical property P is associated with a subset $\mathcal{P}$ consisting of those points in the classical phase space for which the property is true. The points of the phase space which do not belong to $\mathcal{P}$ form the *complementary set* $\sim\mathcal{P}$, and this complementary set defines the negation "NOT P" of the property P. We shall write it as $\sim P$ or as $\tilde{P}$. Alternatively, one can define $\tilde{P}$ as the property which is true if and only if P is false, and false if and only if P is true. From this as well as from the other definition it is obvious that the the negation of the negation of a property is the same as the original property: $\sim\tilde{P}$ or $\sim(\sim P)$ is the same property as P. The indicator $\tilde{P}(\gamma)$ of the property $\sim P$, see (4.1), is given by the formula

$$\tilde{P} = I - P, \tag{4.35}$$

or $\tilde{P}(\gamma) = I(\gamma) - P(\gamma)$, where $I(\gamma)$, the indicator of the identity property, is equal to 1 for all values of γ. Thus $\tilde{P}$ is equal to 1 (true) if P is 0 (false), and $\tilde{P} = 0$ when $P = 1$.

Once again consider Fig. 2.1 on page 9, the phase space of a one-dimensional harmonic oscillator, where the ellipse corresponds to an energy E_0. The property P that the energy is less than or equal to E_0 corresponds to the set $\mathcal{P}$ of points inside and on the ellipse. Its negation $\tilde{P}$ is the property that the energy is greater than E_0, and the corresponding region $\sim\mathcal{P}$ is all the points outside the ellipse. The vertical band $\mathcal{Q}$ corresponds to the property Q that the position of the particle is in the interval $x_1 \leq x \leq x_2$. The negation of Q is the property $\tilde{Q}$ that the particle lies outside this interval, and the corresponding set of points $\sim\mathcal{Q}$ in the phase space consists of the half planes to the left of $x = x_1$ and to the right of $x = x_2$.

A property of a quantum system is associated with a subspace of the Hilbert space, and thus the negation of this property will also be associated with some subspace of the Hilbert space. Consider, for example, a toy model with $M_a = 2 = M_b$. Its Hilbert space consists of all linear combinations of the states $|-2\rangle$, $|-1\rangle$, $|0\rangle$, $|1\rangle$, and $|2\rangle$. Suppose that P is the property associated with the projector

$$P = [0] + [1] \tag{4.36}$$

projecting onto the subspace $\mathcal{P}$ of all linear combinations of $|0\rangle$ and $|1\rangle$. Its physical interpretation is that the quantum particle is confined to these two sites, i.e., it is not at some location apart from these two sites. The negation $\tilde{P}$ of P is the property that the particle is not confined to these two sites, but is instead someplace else, so the corresponding projector is

$$\tilde{P} = [-2] + [-1] + [2]. \tag{4.37}$$

This projects onto the orthogonal complement $\mathcal{P}^\perp$ of $\mathcal{P}$, see Sec. 3.4, consisting of all linear combinations of $|-2\rangle$, $|-1\rangle$ and $|2\rangle$. Since the identity operator for this Hilbert space is given by

$$I = \sum_{m=-2}^{2} [m], \tag{4.38}$$

see (3.52), it is evident that

$$\tilde{P} = I - P. \tag{4.39}$$

This is precisely the same as (4.35), except that the symbols now refer to quantum projectors rather than to classical indicators.

As a second example, consider a one-dimensional harmonic oscillator, Sec. 4.4. Suppose that P is the property that the energy is less than or equal to 2 in units of $\hbar\omega$. The corresponding projector is

$$P = [\phi_0] + [\phi_1] \tag{4.40}$$

in the notation used in Sec. 4.3. The negation of P is the property that the energy is greater than 2, and its projector is

$$\tilde{P} = [\phi_2] + [\phi_3] + [\phi_4] + \cdots = I - P. \tag{4.41}$$

In this case, P projects onto a finite and $\tilde{P}$ onto an infinite-dimensional subspace of $\mathcal{H}$.

As a third example, consider the property X that a particle in one dimension is located in (i.e., not outside) the interval (4.19), $x_1 \leq x \leq x_2$; the corresponding projector X was defined in (4.20). Using the fact that $I\psi(x) = \psi(x)$, it is easy to show that the projector $\tilde{X} = I - X$, corresponding to the property that the particle is located outside (not inside) the interval (4.41) is given by

$$\tilde{X}\psi(x) = \begin{cases} 0 & \text{for } x_1 \leq x \leq x_2, \\ \psi(x) & \text{for all other } x \text{ values.} \end{cases} \tag{4.42}$$

(Note that in this case the action of the projectors X and $\tilde{X}$ is to multiply $\psi(x)$ by the indicator function for the corresponding classical property.)

As a final example, consider a spin-half particle, and let P be the property $S_z = +1/2$ (in units of $\hbar$) corresponding to the projector $[z^+]$. One can think of this as analogous to a toy model with $M = 2$ sites $m = 0, 1$, where $[z^+]$ corresponds to $[0]$. Then it is evident from the earlier discussion that the negation $\tilde{P}$ of P will be the projector $[z^-]$, the counterpart of $[1]$ in the toy model, corresponding to the property $S_z = -1/2$. Of course, the same reasoning can be applied with z replaced by an arbitrary direction w: The property $S_w = -1/2$ is the negation of $S_w = +1/2$, and vice versa.

The relationship between the projector for a quantum property and the projector for its negation, (4.39), is formally the same as the relationship between the corresponding indicators for a classical property, (4.35). Despite this close analogy, there is actually an important difference. In the classical case, the subset $\sim\mathcal{P}$ corresponding to $\tilde{P}$ is the complement of the subset corresponding to P: any point in the phase space is in one or the other, and the two subsets do not overlap. In the quantum case, the subspaces $\mathcal{P}^\perp$ and $\mathcal{P}$ corresponding to $\tilde{P}$ and P have one element in common, the zero vector. This is different from the classical phase space, but is not important, for the zero vector by itself stands for the property which is always false, corresponding to the empty subset of the classical phase space. Much more significant is the fact that $\mathcal{H}$ contains many nonzero elements which belong neither to $\mathcal{P}^\perp$ nor to $\mathcal{P}$. In particular, the sum of a nonzero vector from $\mathcal{P}^\perp$ and a nonzero vector from $\mathcal{P}$ belongs to $\mathcal{H}$, but does not belong to either of these subspaces. For example, the ket $|x^+\rangle$ for a spin-half particle corresponding to $S_x = +1/2$ belongs neither to the subspace associated with $S_z = +1/2$ nor to that of its negation $S_z = -1/2$. Thus despite the formal parallel, the difference between the mathematics of Hilbert space and that of a classical phase space means that negation is not quite the same thing in quantum physics as it is in classical physics.

4.5 Conjunction and Disjunction (AND, OR)

Consider two different properties P and Q of a classical system, corresponding to subsets $\mathcal{P}$ and $\mathcal{Q}$ of its phase space. The system will possess both properties simultaneously if its phase point γ lies in the intersection $\mathcal{P}\cap\mathcal{Q}$ of the sets $\mathcal{P}$ and $\mathcal{Q}$ or, using indicators, if $P(\gamma) = 1 = Q(\gamma)$. See the Venn diagram in Fig. 4.1(a). In this case we can say that the system possesses the property "P AND Q", the *conjunction* of P and Q, which can be written compactly as $P \wedge Q$. The corresponding indicator function is

$$P \wedge Q = PQ, \tag{4.43}$$

that is, $(P \wedge Q)(\gamma)$ is the function $P(\gamma)$ times the function $Q(\gamma)$. In the case of a one-dimensional harmonic oscillator, let P be the property that the energy is less than E_0, and Q the property that x lies between x_1 and x_2. Then the indicator PQ for the combined property $P \wedge Q$, "energy less than E_0 AND x between x_1 and x_2", is 1 at those points in the crosshatched band in Fig. 2.1 which lie inside the ellipse, and 0 everywhere else.

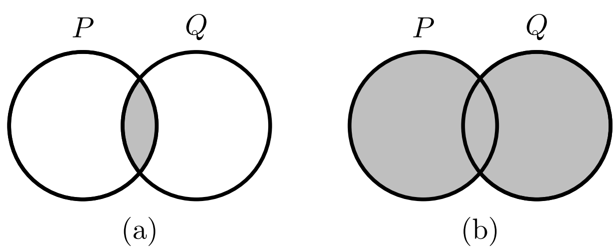

Figure 4.1: The circles represent the properties P and Q. In (a) the grey region is $P \wedge Q$, and in (b) it is $P \vee Q$.

Given the close correspondence between classical indicators and quantum projectors, one might expect that the projector for the quantum property $P \wedge Q$ (P AND Q) would be the product of the projectors for the separate properties, as in (4.43). This is indeed the case *if P and Q commute with each other,* that is, if

$$PQ = QP. \tag{4.44}$$

In this case it is easy to show that the product PQ is a projector satisfying the two conditions in (3.34). On the other hand, if (4.44) is not satisfied, then PQ will not be a Hermitian operator, so it cannot be a projector. In this section we will discuss the conjunction and disjunction of properties P and Q *assuming that the two projectors commute.* The case in which they do not commute is taken up in Sec. 4.6.

As a first example, consider the case of a one-dimensional harmonic oscillator in which P is the property that the energy E is less than 3 (in units of $\hbar\omega$), and Q the property that E is greater than 2. The two projectors are

$$P = [\phi_0] + [\phi_1] + [\phi_2], \quad Q = [\phi_2] + [\phi_3] + [\phi_4] + \cdots, \tag{4.45}$$

and their product is $PQ = QP = [\phi_2]$, the projector onto the state with energy 2.5. As this is the only possible energy of the oscillator which is both greater than 2 and less than 3, the result makes sense.

As a second example, suppose that the property X corresponds to a quantum particle inside (not outside) the interval (4.19), $x_1 \leq x \leq x_2$, and X' to the property that the particle is inside the interval

$$x_1' \leq x \leq x_2'. \tag{4.46}$$

In addition, assume that the endpoints of these intervals are in the order

$$x_1 < x_1' < x_2 < x_2'. \tag{4.47}$$

For a classical particle, $X \wedge X'$ clearly corresponds to its being inside the interval

$$x_1' \leq x \leq x_2. \tag{4.48}$$

In the quantum case, it is easy to show that $XX' = X'X$ is the projector which when applied to a wave function $\psi(x)$ sets it equal to zero everywhere outside the interval (4.48) while leaving it unchanged inside this interval. This result is sensible, because if a wave packet lies inside the interval (4.48), it will also be inside both of the intervals (4.41) and (4.46).

When two projectors P and Q are mutually orthogonal in the sense defined in Sec. 3.5,

$$PQ = 0 = QP \tag{4.49}$$

(each equality implies the other), the corresponding properties P and Q are *mutually exclusive* in the sense that if one is true, the other must be false. The reason is that the 0 operator which represents the conjunction $P \wedge Q$ corresponds, as does the 0 indicator for a classical system, to the property which is always false. Hence it is impossible for both P and Q to be true at the same time, for then $P \wedge Q$ would be true. As an example, consider the harmonic oscillator discussed earlier, but change the definitions so that P is the property $E < 2$ and Q the property $E > 3$. Then $PQ = 0$, for there is no energy which is both less than 2 and greater than 3. Similarly, if the intervals corresponding to X and X' for a particle in one dimension do not overlap—e.g., suppose $x_2 < x_1'$

in place of (4.47)—then $XX' = 0$, and if the particle is between x_1 and x_2, it cannot be between x'_1 and x'_2. Note that this means that a quantum particle, just like its classical counterpart, can never be in two places at the same time, contrary to some misleading popularizations of quantum theory.

The *disjunction* of two properties P and Q, "P OR Q", where "OR" is understood in the non-exclusive sense of "P or Q or both", can be written in the compact form $P \vee Q$. If P and Q are classical properties corresponding to the subsets $\mathcal{P}$ and $\mathcal{Q}$ of a classical phase space, $P \vee Q$ corresponds to the union $\mathcal{P} \cup \mathcal{Q}$ of these two subsets, see Fig. 4.1(b), and the indicator is given by:

$$P \vee Q = P + Q - PQ, \tag{4.50}$$

where the final term $-PQ$ on the right makes an appropriate correction at points in $\mathcal{P} \cap \mathcal{Q}$ where the two subsets overlap, and $P + Q = 2$.

The notions of disjunction (OR) and conjunction (AND) are related to each other by formulas familiar from elementary logic:

$$\begin{aligned} \sim(P \vee Q) &= \tilde{P} \wedge \tilde{Q}, \\ \sim(P \wedge Q) &= \tilde{P} \vee \tilde{Q}. \end{aligned} \tag{4.51}$$

The negation of the first of these yields

$$P \vee Q = \sim(\tilde{P} \wedge \tilde{Q}), \tag{4.52}$$

and one can use this expression along with (4.35) to obtain the right side of (4.50):

$$I - [(I - P)(I - Q)] = P + Q - PQ. \tag{4.53}$$

Thus if negation and conjunction have already been defined, disjunction does not introduce anything that is really new.

The preceding remarks also apply to the quantum case. In particular, (4.53) is valid if P and Q are projectors. However, $P + Q - PQ$ is a projector if and only if $PQ = QP$. Thus as long as P and Q commute, we can use (4.50) to define the projector corresponding to the property P OR Q. There is, however, something to be concerned about. Suppose, to take a simple example, $P = [0]$ and $Q = [1]$ for a toy model. Then (4.50) gives $[0] + [1]$ for $P \vee Q$. However, as pointed out earlier, the subspace onto which $[0] + [1]$ projects contains kets which do not have either the property P or the property Q. Thus $[0] + [1]$ means something less definite than $[0]$ or $[1]$. A satisfactory resolution of this problem requires the notion of a quantum Boolean event algebra, which will be introduced in Sec. 5.2. In the meantime we will simply adopt (4.50) as a definition of what is meant by the quantum projector $P \vee Q$ when $PQ = QP$, and leave till later a discussion of just how it is to be interpreted.

4.6 Incompatible Properties

The situation in which two projectors P and Q do not commute with each other, $PQ \neq QP$, has no classical analog, since the product of two indicator functions on the classical phase space does not depend upon the order of the factors. Consequently, classical physics gives no guidance as to how to think about the conjunction $P \wedge Q$ (P AND Q) of two quantum properties when their projectors do not commute.

Consider the example of a spin-half particle, let P be the property $S_x = +1/2$, and Q the property that $S_z = +1/2$. The projectors are

$$P = [x^+], \quad Q = [z^+], \tag{4.54}$$

and it is easy to show by direct calculation that $[x^+][z^+]$ is unequal to $[z^+][x^+]$, and that neither is a projector. Let us suppose that it is nevertheless possible to define a property $[x^+] \wedge [z^+]$. To what subspace of the two-dimensional spin space might it correspond? Every one-dimensional subspace of the Hilbert space of a spin-half particle corresponds to the property $S_w = +1/2$ for some direction w in space, as discussed in Sec. 4.2. Thus if $[x^+] \wedge [z^+]$ were to correspond to a one-dimensional subspace, it would have to be associated with such a direction. Clearly the direction cannot be x, for $S_x = +1/2$ does not have the property $S_z = +1/2$; see the discussion in Sec. 4.2. By similar reasoning it cannot be z, and all other choices for w are even worse, because then $S_w = +1/2$ possesses neither the property $S_x = +1/2$ nor the property $S_z = +1/2$, much less both of these properties!

If one-dimensional subspaces are out of the question, what is left? There is a two-dimensional "subspace" which is the entire space, with projector I corresponding to the property which is always true. But given that neither $[x^+]$ nor $[z^+]$ is a property which is always true, it seems ridiculous to suppose that $[x^+] \wedge [z^+]$ corresponds to I. There is also the zero-dimensional subspace which contains only the zero vector, corresponding to the property which is always false. Does it make sense to suppose that $[x^+] \wedge [z^+]$, thought of as a particular property possessed by a given spin-half particle at a particular time, is always false in the sense that there are no circumstances in which it could be true? If we adopt this proposal we will, obviously, also want to say that $[x^+] \wedge [z^-]$ is always false. Following the usual rules of logic, the disjunction (OR) of two false propositions is false. Therefore, the left side of

$$([x^+] \wedge [z^+]) \vee ([x^+] \wedge [z^-]) = [x^+] \wedge ([z^+] \vee [z^-]) = [x^+] \wedge I = [x^+] \tag{4.55}$$

is always false, and thus the right side, the property $S_x = +1/2$, is always false. But this makes no sense, for there are circumstances in which $S_x = +1/2$ is true.

To obtain the first equality in (4.55) requires the use of the distributive identity

$$(P \wedge Q) \vee (P \wedge R) = P \wedge (Q \vee R) \tag{4.56}$$

of standard logic, with $P = [x^+]$, $Q = [z^+]$, and $R = [z^-]$. One way of avoiding the silly result implied by (4.55) is to modify the laws of logic so that the distributive law does not hold. In fact, Birkhoff and von Neumann proposed a special *quantum logic* in which (4.56) is no longer valid. Despite a great deal of effort, this quantum logic has not turned out to be of much help in understanding quantum theory, and we shall not make use of it.

In conclusion, there seems to be no plausible way to assign a subspace to the conjunction $[x^+]\wedge[z^+]$ of these two properties, or to any other conjunction of two properties of a spin-half particle which are represented by non-commuting projectors. Such conjunctions are therefore meaningless in the sense that the Hilbert space approach to quantum theory, in which properties are associated with subspaces, cannot assign them a meaning. It is sometimes said that it is impossible to *measure* both S_x and S_z simultaneously for a spin-half particle. While this statement is true, it is important to note that the inability to carry out such a measurement reflects the fact that there is no corresponding property which could be measured. How could a measurement tell us, for example, that for a spin-half particle $S_x = +1/2$ and $S_z = +1/2$, if the property $[x^+] \wedge [z^+]$ cannot even be defined?

Guided by the spin-half example, we shall say that two properties P and Q of any quantum system are *incompatible* when their projectors do not commute, $PQ \neq QP$, and that the conjunction $P \wedge Q$ of incompatible properties is meaningless in the sense that quantum theory assigns it no meaning. On the other hand, if $PQ = QP$, the properties are *compatible*, and their conjunction $P \wedge Q$ corresponds to the projector PQ.

To say that $P \wedge Q$ is *meaningless* when $PQ \neq QP$ is very different from saying that it is *false*. The negation of a false statement is a true statement, so if $P \wedge Q$ is false, its negation $\tilde{P} \vee \tilde{Q}$, see (4.51), is true. On the other hand, the negation of a meaningless statement is equally meaningless. Meaningless statements can also occur in ordinary logic. Thus if P and Q are two propositions of an appropriate sort, $P \wedge Q$ is meaningful, but $P \wedge \vee\, Q$ is meaningless: this last expression cannot be true or false, it just doesn't make any sense. In the quantum case, "$P \wedge Q$" when $PQ \neq QP$ is something like $P \wedge \vee\, Q$ in ordinary logic. Books on logic always devote some space to the rules for constructing meaningful statements. Physicists when reading books on logic tend to skip over the chapters which give these rules, because the rules seem intuitively obvious. In quantum theory, on the other hand, it is necessary to pay some attention to the rules which separate meaningful and meaningless statements, because they are not the same as in classical physics, and hence they are not intuitively obvious, at least until one has built up some intuition for the quantum world.

When P and Q are incompatible, it makes no sense to ascribe both properties to a single system at the same instant of time. However, this does not exclude the possibility that P might be a meaningful (true or false) property at one instant of time and Q a meaningful property at a *different* time. We will discuss the time dependence of quantum systems starting in Ch. 7. Similarly, P and Q might refer to two distinct physical systems: for example, there is no problem in supposing that $S_x = +1/2$ for one spin-half particle, and $S_z = +1/2$ for a *different* particle.

At the end of Sec. 4.1 we stated that if a quantum system is described by a ket $|\psi\rangle$ which is not an eigenstate of a projector P, then the physical property associated with this projector is undefined. The situation can also be discussed in terms of incompatible properties, for saying that a quantum system is described by $|\psi\rangle$ is equivalent to asserting that it has the property $[\psi]$ corresponding to the ray which contains $|\psi\rangle$. It is easy to show that the projectors $[\psi]$ and P commute if and only if $|\psi\rangle$ is an eigenstate of P, whereas in all other cases $[\psi]P \neq P[\psi]$, so they represent incompatible properties.

It is possible for $|\psi\rangle$ to simultaneously be an eigenstate with eigenvalue 1 of two incompatible projectors P and Q. For example, for the toy model of Sec. 4.2, let

$$|\psi\rangle = |2\rangle, \quad P = [\sigma] + [2], \quad Q = [1] + [2], \tag{4.57}$$

where $|\sigma\rangle$ is defined in (4.10). The definition given in Sec. 4.1 allows us to conclude that the quantum system described by $|\psi\rangle$ has the property P, but we could equally well conclude that it has the property Q. However, it makes no sense to say that it has both properties. Sorting out this issue will require some additional concepts found in later chapters.

If the conjunction of incompatible properties is meaningless, then so is the disjunction of incompatible properties: $P \vee Q$ (P OR Q) makes no sense if $PQ \neq QP$. This follows at once from (4.52), because if P and Q are incompatible, so are their negations $\tilde{P}$ and $\tilde{Q}$, as can be seen by multiplying out $(I-P)(I-Q)$ and comparing it with $(I-Q)(I-P)$. Hence $\tilde{P} \wedge \tilde{Q}$ is meaningless, and so is its negation. Other sorts of logical comparisons, such as the exclusive OR (XOR), are also not possible in the case of incompatible properties.

If $PQ \neq QP$, the question "Does the system have property P or does it have property Q?" makes no sense if understood in a way which requires a comparison of these two incompatible properties. Thus one answer might be, "The system has property P but it does not have property Q". This is equivalent to affirming the truth of P and the falsity of Q, so that P and $\tilde{Q}$ are simultaneously true. But since $P\tilde{Q} \neq \tilde{Q}P$, this makes no sense. Another answer might be that "The system has both properties P and Q", but the assertion that P and Q are simultaneously true also does not make sense. And a question to which one cannot give a meaningful answer is not a meaningful question.

In the case of a spin-half particle it does not make sense to ask whether $S_x = +1/2$ or $S_z = +1/2$, since the corresponding projectors do not commute with each other. This may seem surprising, since it is possible to set up a device which will produce spin-half particles with a definite polarization, $S_w = +1/2$, where w is a direction determined by some property or setting of the device. (This could, for example, be the direction of the magnetic field gradient in a Stern-Gerlach apparatus, Sec. 17.2.) In such a case one can certainly ask whether the setting of the device is such as to produce particles with $S_x = +1/2$ or with $S_z = +1/2$. However, the values of components of spin angular momentum for a particle polarized by this device are then properties *dependent* upon properties of the device in the sense described in Ch. 14, and can only sensibly be discussed with reference to the device.

Along with different components of spin for a spin-half particle, it is easy to find many other examples of incompatible properties of quantum systems. Thus the projectors X and P in Sec. 4.3, for the position of a particle to lie between x_1 and x_2 and its momentum between p_1 and p_2, respectively, do not commute with each other. In the case of a harmonic oscillator, neither X nor P commutes with projectors, such as $[\phi_0] + [\phi_1]$, which define a range for the energy. That quantum operators, including the projectors which represent quantum properties, do not always commute with each other is a consequence of employing the mathematical structure of a quantum Hilbert space rather than that of a classical phase space. Consequently, there is no way to get around the fact that quantum properties cannot always be thought of in the same way as classical properties. Instead, one has to pay attention to the rules for combining them if one wants to avoid inconsistencies and paradoxes.